

Chapter 3

**INTERSECTION OF BLACK HOLE THEORY AND
QUANTUM CHROMODYNAMICS: THE GLUON
PROPAGATOR CORRESPONDING TO LINEAR
CONFINEMENT AT LARGE DISTANCES AND
RELATIVISTIC BOUND STATES IN THE CONFINING
SU(N)-YANG-MILLS FIELDS**

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Abstract

The exact nonperturbative confining solutions of the SU(3)-Yang-Mills equations recently obtained by author in Minkowski spacetime with the help of the black hole theory techniques are analysed and on the basis of them the gluon propagator corresponding to linear confinement at large distances (small momenta) is constructed in a nonperturbative way. At small distances (large momenta) the resulting propagator passes on to the standard (nonperturbative) gluon propagator used in the perturbative quantum chromodynamics (QCD). The results suggest some scenario of linear confinement for mesons and quarkonia which is also outlined. As a consequence there arises a motivation for studying the relativistic bound states in the above confining SU(N)-Yang-Mills fields. This possibility is realized for $N = 2, 3, 4$ with the aid of the black hole theory results about spinor fields on black holes with a subsequent application to the charmonium spectrum in the most important physical case $N = 3$. Incidentally uniqueness of the confining solutions is discussed and a comparison with the nonrelativistic potential approach is given.

Contents

1 Introduction and Preliminary Remarks

3

2	Confining Solutions of the Maxwell and SU(3)-Yang-Mills Equations	5
2.1	Black Hole Theory Techniques	5
2.2	Electrodynamics	5
2.3	SU(3)-Yang-Mills Theory	6
3	Linear Confinement in QED	7
3.1	Standard Photon Propagator	7
3.2	Electrodynamics with Linear Confinement	9
3.3	Momentum Representation	10
4	Linear Confinement in QCD	11
4.1	Motivation	11
4.2	A Scenario for Linear Confinement	13
5	Confining Solutions of the SU(N)-Yang-Mills Equations	14
5.1	Motivation	14
5.2	Role of Diagonal Gauge	17
5.3	U(1)-case	19
5.4	$N = 2$	20
5.5	$N = 3$	21
5.6	$N = 4$	21
6	Spectrum of Bound States in the Coulomb-Like Case	22
6.1	Results from the Black Hole Theory about Eigenspinors of the (Twisted) Euclidean Dirac Operator on \mathbb{S}^2	
6.2	U(1)-case	24
6.3	$N = 2$	27
6.4	$N = 3$	28
6.5	$N = 4$	29
7	Spectrum of Bound States in the Coulomb-Linear Case	30
7.1	U(1)-case	30
7.2	$N = 2$	32
7.3	$N = 3$	33
7.4	$N = 4$	34
7.5	Remark about the case $N > 4$	35
7.6	Nonrelativistic Limit	36
8	Uniqueness of the Confining Solutions	36
8.1	Uniqueness	36
8.1.1	Remark Concerning the Wilson Confinement Criterion	38
8.2	Nonrelativistic Confining Potentials	38
8.2.1	Maxwell Equations	39
8.2.2	SU(3)-Yang-Mills Equations	39

8.3	Remark about Search for Nonrelativistic Confining Potentials	40
9	Application to the Charmonium Spectrum	40
9.1	Relativistic Spectrum of Charmonium	40
9.2	Nonrelativistic Limit	42
9.3	Electromagnetic Transitions	42
9.4	Estimates of Gluon Concentrations and Magnetic Colour Field Sstrength . .	42
9.5	Numerical Results and Concluding Remarks	45
10	Conclusion	46
11	Appendix A: Hodge Star Operator $*$ on MinkowskiSpacetime in Spherical Coordinates	48
12	Appendix B: $SU(N)$-Lie Algebras and their CartanSubalgebras	49
13	Appendix C: Eigenspinors of the Euclidean Dirac Operator on \mathbb{S}^2 at $\lambda = \pm 1$	52
14	Appendix D: Condition $\sigma_2 \Phi \approx \sin \vartheta \Phi$	53
15	Appendix E: More General Ansatz	53

1 Introduction and Preliminary Remarks

As soon as quantum chromodynamics (QCD) was proposed as the main candidate for the theory of strong interactions [1] at once there arised the question about the confinement of quarks within the framework of QCD. In the late seventies of XX century the main approaches to solve the problem were formed (see, e.g., review of Ref. [2]) and they actually remain the same ones up to now. For this purpose miscellaneous techniques were elaborated, for example, strong coupling expansions, lattice approach, instanton improvement of perturbation theory, nonrelativistic potential approach and so on. It should be noted, however, none of the mentioned various directions has so far led to a generally accepted theory of quark confinement.

In this paper we would like from another side to have analysed one of the possible approaches. The question is about a nonperturbative modification of gluon propagator which might correspond to linear confinement between quarks at large distances. The very simple idea of modifying the mentioned propagator arises when considering the naive Fourier transform for the power potentials of form r^λ (for more details, see e.g. Ref. [2]). Then at $\lambda = 1$ (linear confinement) the conforming Fourier transform (propagator) is of order $|\mathbf{k}|^{-4}$ in momentum space, while the case $\lambda = -1$ (Coulomb potential) gives the standard gluon propagator $\sim |\mathbf{k}|^{-2}$. All the attempts to obtain the necessary behaviour, however, for example, by summing a selective infinite set of perturbation diagrams with using the Dyson–Schwinger equations for the propagator failed [2]. It is clear why: it is impossible to get anything nonperturbative like confinement by perturbative techniques. Some new

possibilities in this direction were connected with lattice theories (for more details, see, e.g., Ref. [3] and references therein) but the results here are mainly of qualitative character.

To our mind, from the very outset the problem should be considered on the basis of the exact nonperturbative solutions of the SU(3)-Yang-Mills equations modelling quark confinement which, in what follows, we shall call *the confining solutions*. Such solutions will be supposed to contain only the components of the SU(3)-field which are Coulomb-like or linear in r , the distance between quarks. In Ref. [4] a number of such solutions have been obtained and the corresponding spectrum of Dirac equation describing the relativistic bound states in those confining SU(3)-Yang-Mills fields has been analysed. Further in Refs. [5, 6, 35] the results obtained were successfully applied to the description of the quarkonia spectra (charmonium and bottomonium). In its turn, the mentioned description suggests that linear confinement is (classically) governed by the magnetic colour field linear in r and, as was mentioned in Refs. [5, 31], one can try to modify the gluon propagator nonperturbatively at quantum level for to generate the mentioned magnetic colour field at classical level. One part of the present paper is just devoted to it. It should be noted, however, that all the main features of such a modification may occur already within quantum electrodynamics (QED) that should be not surprising because, as is historically known (see, e.g., Ref. [7]), the standard gluon propagator is in fact the slightly modified photon one of QED. Under the circumstances we shall conduct our considerations mainly within QED but incidentally making remarks to generalize the results obtained to the QCD case. Considerations of the first part of paper (Sections 1–4) inevitably lead to the task of a more thorough analysis of the Yang–Mills and Dirac equations derived from QCD lagrangian which is realized in the rest of paper.

Let us introduce some notations. Further we shall deal with the metric of the flat Minkowski spacetime M that we write down [using the ordinary set of local rectangular (Cartesian) (x, y, z) or spherical (r, ϑ, φ) coordinates for spatial part] in the forms

$$ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu \equiv dt^2 - dx^2 - dy^2 - dz^2 \equiv dt^2 - dr^2 - r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad (1.1)$$

so the components $g_{\mu\nu}$ take different values depending on the choice of coordinates. Besides we have $\delta = |\det(g_{\mu\nu})| = (r^2 \sin \vartheta)^2$ in spherical coordinates and the exterior differential $d = \partial_t dt + \partial_x dx + \partial_y dy + \partial_z dz$ or $d = \partial_t dt + \partial_r dr + \partial_\vartheta d\vartheta + \partial_\varphi d\varphi$ in the corresponding coordinates. We denote 3-dimensional vectors by bold font so $X = (t, x, y, z) \equiv (t, \mathbf{r}), k = (k_0, k_1, k_2, k_3) \equiv (k_0, \mathbf{k})$ with $r^2 = x^2 + y^2 + z^2, |\mathbf{k}| = \sqrt{k_1^2 + k_2^2 + k_3^2}$. The Fourier transform $\tilde{\Phi}(k)$ of some function $\Phi(X)$ is formally defined by the relations ($kX = k_0 t - k_1 x - k_2 y - k_3 z$)

$$\begin{aligned} \tilde{\Phi}(k) &= \int_M \exp(ikX) \Phi(X) d^4 X = F[\Phi], \\ \Phi(X) &= \frac{1}{(2\pi)^4} \int_M \exp(-ikX) \tilde{\Phi}(k) d^4 k \end{aligned} \quad (1.2)$$

but it is treated in the sense of the theory of generalized functions (distributions) (see, e.g., Refs. [9]) and we denote $d^4 k = dk_0 dk_1 dk_2 dk_3, d^4 X = dt dV, dV = dx dy dz$ or

$dV = \sqrt{\delta} dr d\vartheta d\varphi$ while for the generalized δ -functions we use the notations $\delta(X) \equiv \delta(t)\delta(x)\delta(y)\delta(z)$, $\delta(\mathbf{r}) \equiv \delta(x)\delta(y)\delta(z)$. Other mathematical results necessary for our considerations are gathered in Appendices *A, B, C, D, E*.

Throughout the paper we employ the system of units with $\hbar = c = 1$, unless explicitly stated otherwise. Finally, we shall denote $L_2(F)$ the set of the modulo square integrable complex functions on any manifold F furnished with an integration measure while $L_2^n(F)$ will be the n -fold direct product of $L_2(F)$ endowed with the obvious scalar product.

2 Confining Solutions of the Maxwell and SU(3)-Yang-Mills Equations

2.1 Black Hole Theory Techniques

For obtaining a set of the confining solutions within the given paper we shall employ the techniques used in Refs. [8] for finding the $U(N)$ -monopole solutions in black hole physics and the essence of those techniques consists in systematic usage of the Hodge star operator (see Appendix *A*) conforming to metric (1.1). As is known, such a metric can be obtained from the Schwarzschild black hole metric when the black hole mass is equal to 0.

Really, if writing down the Yang-Mills equations (B.3) in components then we shall be drowned in a sea of indices which will strongly hamper searching for one or another ansatz and make it practically immense. Using the Hodge star operator as well as the rules of external calculus makes the problem quite foreseeable and quickly leads to the aim.

As was remarked in Appendix *B*, the sought solutions are usually believed to obey an additional condition and as the latter one in the present paper we take the Lorentz condition that can be written in the form

$$\text{div}(A) = 0, \quad (2.1)$$

where the divergence of the Lie algebra valued 1-form $A = A_\mu^a T_a dx^\mu$ is defined by the relation

$$\text{div}(A) = \frac{1}{\sqrt{\delta}} \partial_\mu (\sqrt{\delta} g^{\mu\nu} A_\nu). \quad (2.2)$$

2.2 Electrodynamics

We proceed from the second pair of Maxwell equations (B.5)

$$d * F = J \quad (2.3)$$

with $F = dA$, $A = A_\mu dx^\mu$ and the Hodge star operator $*$ is defined, for example, on 2-forms $F = F_{\mu\nu} dx^\mu \wedge dx^\nu$ in Minkowski spacetime M provided with a pseudoriemannian metric $g_{\mu\nu}$ (1.1) by the relation (see Appendix *A*)

$$F \wedge *F = (g^{\mu\alpha} g^{\nu\beta} - g^{\mu\beta} g^{\nu\alpha}) F_{\mu\nu}^a F_{\alpha\beta}^a \sqrt{\delta} dx^1 \wedge dx^2 \cdots \wedge dx^4 \quad (2.4)$$

in local coordinates $x = (x^\mu)$ while $J = j_\mu * (dx^\mu)$ with a 4-dimensional electromagnetic density current $j = j_\mu dx^\mu$. Let $J = 0$ and we shall search for the solution of (2.3) in the form $A = A_t(r)dt + A_\varphi(r)d\varphi$. It is then easy to check that $F = dA = -\partial_r A_t dt \wedge dr + \partial_r A_\varphi dr \wedge d\varphi$ and since $*(dt \wedge dr) = -r^2 \sin \vartheta d\vartheta \wedge d\varphi$, $*(dr \wedge d\varphi) = -\frac{1}{\sin \vartheta} dt \wedge d\vartheta$ we get $*F = r^2 \sin \vartheta \partial_r A_t d\vartheta \wedge d\varphi - \frac{1}{\sin \vartheta} \partial_r A_\varphi dt \wedge d\vartheta$. From here it follows that Eq. (2.3) yields

$$\partial_r(r^2 \partial_r A_t) = 0, \quad \partial_r^2 A_\varphi = 0, \quad (2.5)$$

and we write down the solutions of (2.5) as

$$A_t = \frac{a}{r} + A, \quad A_\varphi = br + B \quad (2.6)$$

with some constants a, b, A, B parametrizing solutions (further for the sake of simplicity let us put $a = 1, b = 1$ GeV, $A = B = 0$).

To interpret solutions (2.6) in the more habitual physical terms let us pass on to Cartesian coordinates employing the relations

$$\varphi = \arctan(y/x), \quad d\varphi = \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy \quad (2.7)$$

which entails

$$A_\varphi d\varphi = -\frac{ry}{x^2 + y^2} dx + \frac{rx}{x^2 + y^2} dy \quad (2.8)$$

and we conclude that the solutions of (2.6) describe the combination of the electric Coulomb field with potential $\Phi = 1/r$ and the constant magnetic field with vector-potential

$$\mathbf{A} = (A_x, A_y, A_z) = \left(-\frac{ry}{x^2 + y^2}, \frac{rx}{x^2 + y^2}, 0 \right) = \left(-\frac{\sin \varphi}{\sin \vartheta}, \frac{\cos \varphi}{\sin \vartheta}, 0 \right), \quad (2.9)$$

which is *linear* in r in spherical coordinates and the 3-dimensional divergence $\text{div} \mathbf{A} = 0$, as can be checked directly. Then Eq. (2.3) in Cartesian coordinates takes the form

$$\Delta \Phi = 0, \quad \text{rot} \text{rot} \mathbf{A} = \Delta \mathbf{A} = 0 \quad (2.10)$$

with the Laplace operator $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$. At last it is easy to check that the solution under consideration satisfies the Lorentz condition (2.2)

$$\text{div}(A) = \frac{1}{\sqrt{\delta}} \partial_\mu (\sqrt{\delta} g^{\mu\nu} A_\nu) = 0. \quad (2.11)$$

2.3 SU(3)-Yang-Mills Theory

Now Eq. (2.3) should be replaced by the Yang-Mills equations (B.3)

$$d * F = g(*F \wedge A - A \wedge *F) + J \quad (2.12)$$

for SU(3)-field $A = A_\mu dx^\mu$, $A_\mu = A_\mu^a T_a$ where the matrices T_a form a basis of the Lie algebra of SU(3) in 3-dimensional space, $a = 1, \dots, 8$ and further let us put $T_a = \lambda_a$, where

λ_a are the Gell-Mann matrices (see Appendix B). After this we search (at $J = 0$) for the solution of (2.12) in the form $A = A_t(r)dt + A_\varphi(r)d\varphi$ with $A_{t,\varphi} = A_{t,\varphi}^3\lambda_3 + A_{t,\varphi}^8\lambda_8$. Evaluating $dF = dA + gA \wedge A$ it is easy to check that the right-hand side of (2.12) is equal to zero and to gain the sought solution in the form (which reflects the fact that for any matrix \mathcal{T} from SU(3)-Lie algebra we have $\text{Tr } \mathcal{T} = 0$)

$$\begin{aligned} A_t^3 + \frac{1}{\sqrt{3}}A_t^8 &= -\frac{a_1}{r} + A_1, -A_t^3 + \frac{1}{\sqrt{3}}A_t^8 = -\frac{a_2}{r} + A_2, -\frac{2}{\sqrt{3}}A_t^8 = \frac{a_1 + a_2}{r} - (A_1 + A_2), \\ A_\varphi^3 + \frac{1}{\sqrt{3}}A_\varphi^8 &= b_1r + B_1, -A_\varphi^3 + \frac{1}{\sqrt{3}}A_\varphi^8 = b_2r + B_2, -\frac{2}{\sqrt{3}}A_\varphi^8 = -(b_1 + b_2)r - (B_1 + B_2), \end{aligned} \quad (2.13)$$

where real constants a_j, A_j, b_j, B_j parametrize the solution, and we wrote down the solution in the combinations that are just needed to insert into the corresponding Dirac equation (see Section 5). From here it follows

$$\begin{aligned} A_t^3 &= [(a_2 - a_1)/r + A_1 - A_2]/2, A_t^8 = [A_1 + A_2 - (a_1 + a_2)/r]\sqrt{3}/2, \\ A_\varphi^3 &= [(b_1 - b_2)r + B_1 - B_2]/2, A_\varphi^8 = [(b_1 + b_2)r + B_1 + B_2]\sqrt{3}/2 \end{aligned} \quad (2.14)$$

and practically the same considerations as the above ones in electrodynamics show that the given solution describes the configuration of the electric Coulomb-like colour field (components A_t) with potentials Φ^3, Φ^8 and the constant magnetic colour field (components A_φ) with vector-potentials $\mathbf{A}^3, \mathbf{A}^8$ which are *linear* in r in spherical coordinates with 3-dimensional divergences $\text{div} \mathbf{A}^3 = \text{div} \mathbf{A}^8 = 0$ while the Eq. (2.12) is easily transformed into the Eqs. (2.10) with an obvious modification. It is also simple to check that the solution under consideration satisfies the Lorentz condition (2.2).

3 Linear Confinement in QED

Now we can investigate how the photon propagator should be modified if in the real world the interaction between two charged particles would not be classically described only by the Coulomb law but it would also include a constant magnetic field linear in r , the distance between particles, so the given field would obey the Maxwell equations. Let us briefly recall the scheme in accordance with that the standard photon propagator is obtained (see, e.g., Ref. [7]).

3.1 Standard Photon Propagator

In Cartesian coordinates the Eq. (2.3) for $A = A_\mu dx^\mu$ at $J = 0$ with Lorentz condition $\text{div}(A) = 0$ reduces to the system

$$\square A_\mu = 0 \quad (3.1)$$

with the d'Alembert operator $\square = \partial_t^2 - \Delta$. Then one constructs the fundamental solution (a Green function) of the system (3.1) (the photon propagator at classical level) as the matrix with elements $D_{\mu\nu}(X)$ obeying the system

$$g_{\sigma\mu} \square D_{\mu\nu}(X) = -g_{\sigma\nu} \delta(X) . \quad (3.2)$$

Further one uses the fundamental solution of the d'Alembert operator [i.e., the solution of the equation $\square K = \delta(X)$] in the form

$$K(X) = \frac{1}{4\pi^2 i (X^2 - i0)} \quad (3.3)$$

with quadratic form $X^2 = t^2 - r^2$ (the exact definitions concerning the generalized functions connected with quadratic forms can be found in Refs. [9]), so that

$$D_{\mu\nu}(X) = -g_{\mu\nu} K(X) = -\frac{g_{\mu\nu}}{4\pi^2 i (X^2 - i0)} . \quad (3.4)$$

After this the photon propagator at quantum level is obtained as the Fourier transform for $D_{\mu\nu}(X)$, namely

$$\tilde{D}_{\mu\nu}(k) = F[D_{\mu\nu}(X)] = -\frac{g_{\mu\nu}}{k^2 + i0} , \quad (3.5)$$

since $\tilde{K} = F[K] = 1/(k^2 + i0)$ with quadratic form $k^2 = k_0^2 - k_1^2 - k_2^2 - k_3^2$ (for more details see Refs. [9]). There arises the question why among a large set of the mathematically possible fundamental solutions (Green functions) for Eq. (3.2) one chooses just the one of (3.4). The answer can be based only on physical considerations.

Indeed, let us take a point particle with a charge e moving with a velocity $\mathbf{v} = \mathbf{v}(t)$. Then, as is known (see, e.g. Ref. [10]), the 4-dimensional density current of such an object is $j = j_\mu dx^\mu = e\delta(\mathbf{r})(dt + \mathbf{v}d\mathbf{r})$, $d\mathbf{r} = (dx, dy, dz)$. Under the circumstances the particle will generate the electromagnetic field with potential $A = A_\mu dx^\mu$ which should be obtained by the contraction of j with a fundamental solution $D_{\mu\nu}(X)$ of Eq. (2.3) or, that is the same, of Eq. (3.1), namely

$$A_\mu(X) = 4\pi \int D_{\mu\nu}(X - X') j_\nu(X') d^4 X' . \quad (3.6)$$

When choosing $D_{\mu\nu}(X)$ equal to that of (3.4) we obtain, for example, for electric potential of the field generated [with replacing $t \rightarrow it$ in the integral over t in (3.6)]

$$A_t(X) = \Phi(X) = \frac{e}{\pi} \int_{-\infty}^{\infty} \frac{dt}{t^2 + r^2} = \frac{e}{r} , \quad (3.7)$$

that is, the Coulomb law. Analogously, other diagonal components of $D_{\mu\nu}(X)$ of (3.4) give, for example, at $\mathbf{v} = \text{const}$ the vector-potential \mathbf{A} for a Coulomb-like magnetic field generated by the point charged particle when its moving (for more details see Ref. [10]). All of that corresponds to experimental data and, as a result, the choice of $D_{\mu\nu}(X)$ in the form (3.4) reflects the real situation in our world.

3.2 Electrodynamics with Linear Confinement

Let us now explore how the photon propagator should be modified if experiment would say to us that when its moving a point charged particle generates a constant magnetic field linear in r , the distance from the particle, additionally to the mentioned Coulomb-like fields.

Under the circumstances we should use the property of any fundamental solution that the latter is determined only to within adding any solution of the conforming homogeneous equation. As we have seen above, components of \mathbf{A} from (2.9) are the solutions of the Laplace equation. On the other hand, as is known, the Coulomb potential is also the solution of the Laplace equation at $r \neq 0$ and besides it is a fundamental solution of the Laplace operator [9]. Consequently, for to obtain the necessary modification of 3-dimensional photon propagator we should add components A_x or A_y of (2.9) to the Coulomb fundamental solution for the conforming components of propagator. To pass on to a 4-dimensional propagator let us recall that the Coulomb fundamental solution of the Laplace operator and the fundamental solution (3.3) of the d'Alembert operator are connected by the so-called method of descent (see, e.g., Ref. [11]) which is in essence expressed by the relation

$$4\pi \int_{-\infty}^{\infty} K(X) dt = -\frac{1}{r}, \quad (3.8)$$

so that to obtain the necessary modification of 4-dimensional propagator we should add some suitable solutions of the d'Alembert (wave) equation $\square f(X) = 0$ to $K(X)$ for the corresponding components of propagator (3.4). As the latter ones we should take functions $(tA_x)/(8\pi^2)$ or $(tA_y)/(8\pi^2)$ [where the factor $1/(8\pi^2)$ is introduced for the sake of further convenience] with A_x or A_y of (2.9). The given functions obviously satisfy the d'Alembert equation. Then the sought photon propagator will have the same components as in (3.4) except for the cases $\mu = \nu = x$ or y where the components will be

$$\begin{aligned} D_{xx}(X) &= \frac{1}{4\pi^2 i(X^2 - i0)} + \frac{tA_x}{8\pi^2} = \frac{1}{4\pi^2 i(X^2 - i0)} - \frac{try}{8\pi^2(x^2 + y^2)}, \\ D_{yy}(X) &= \frac{1}{4\pi^2 i(X^2 - i0)} + \frac{tA_y}{8\pi^2} = \frac{1}{4\pi^2 i(X^2 - i0)} + \frac{trx}{8\pi^2(x^2 + y^2)}. \end{aligned} \quad (3.9)$$

Under this situation when its moving a charged particle might generate an additional constant magnetic field according to the relation (3.6). To specify it let us recall that in electrodynamics [10] any constant magnetic field is connected with a finite motion of charged particles. Let us suppose, for instance, that the particle accomplishes a finite motion within a finite (though perhaps large enough) time in such a way that the velocity projections $v_{x,y}(t)$ are some odd functions of time. Then one may consider that

$$\int (t - t') v_{x,y}(t') dt' \sim \int t' v_{x,y}(t') dt' \sim C = \text{const}, \quad (3.10)$$

and according to (3.6) there appears some constant magnetic field $\sim \mathbf{A}$ of (2.9) linear in r , the distance from particle. So indeed under the certain conditions we could observe the mentioned magnetic field corresponding to the propagator described above.

3.3 Momentum Representation

To get the necessary propagator at quantum level we should carry out the Fourier transform of the just found propagator. So long as for any natural m we have (see Refs. [9])

$$F[t^m] = 2\pi(-i)^m \delta^{(m)}(k_0) \quad (3.11)$$

with m -th derivative of δ -function, then really everything reduces to the Fourier transforms for functions A_x, A_y of (2.9). Since the latter ones are only locally integrable the Fourier transforms should be understood in the sense of the theory of generalized functions [9], namely, through analytical continuation of suitable integrals. Let us find, e.g., $F[A_x]$. In accordance with (1.2) we shall have

$$\begin{aligned} F[A_x] &= \int A_x \exp(-i\mathbf{k}\mathbf{r}) dV = \\ &= - \int_0^\infty r^2 dr \int_0^\pi \exp(-ik_3 r \cos \vartheta) d\vartheta \int_0^{2\pi} \exp[-ir \sin \vartheta (k_1 \cos \varphi + k_2 \sin \varphi)] \sin \varphi d\varphi. \end{aligned} \quad (3.12)$$

Using the relation of Ref. [12]

$$\int_0^{2\pi} \exp(a \cos x + b \sin x) \left(\frac{\sin x}{\cos x} \right) dx = \frac{\pi}{\sqrt{a^2 + b^2}} I_1(\sqrt{a^2 + b^2}) \left(\frac{2b}{2a} \right) \quad (3.13)$$

with the modified Bessel function $I_1(z) = -iJ_1(iz)$, where $J_1(z) = -J_1(-z)$ is the standard Bessel function, we can rewrite (3.12) as

$$F[A_x] = \frac{2\pi i k_2}{\sqrt{k_1^2 + k_2^2}} \int_0^\infty r^2 dr \int_0^\pi \exp(-ik_3 r \cos \vartheta) J_1(r \sin \vartheta \sqrt{k_1^2 + k_2^2}) d\vartheta. \quad (3.14)$$

Further replacing $\cos \vartheta = x$ and employing the formula of Ref. [13]

$$\int_0^a \frac{\cos(b\sqrt{a^2 - x^2})}{\sqrt{a^2 - x^2}} J_1(cx) dx = \frac{1}{ac} [\cos(ab) - \cos(a\sqrt{b^2 + c^2})], \quad (3.15)$$

we get

$$F[A_x] = \frac{4\pi i k_2}{k_1^2 + k_2^2} \int_0^\infty r [\cos(k_3 r) - \cos(r|\mathbf{k}|)] dr. \quad (3.16)$$

At last, using the relation with the Euler Γ -function of Ref. [12]

$$\int_0^\infty x^{q-1} \cos(mx) dx = \frac{\Gamma(q)}{m^q} \cos \frac{\pi q}{2}, \quad (3.17)$$

holding true at $0 < q < 1, m > 0$, we analytically continue the right-hand side of (3.17) over all admissible values q, m which permits [at $q = 2$ with $\Gamma(2) = 1$] to write down

$$F[A_x] = -\frac{4\pi i k_2}{k_1^2 + k_2^2} \left[\frac{1}{k_3^2} - \frac{1}{|\mathbf{k}|^2} \right] = -\frac{4\pi i k_2}{|\mathbf{k}|^2 k_3^2}. \quad (3.18)$$

Analogous consideration yields

$$F[A_y] = \frac{4\pi i k_1}{|\mathbf{k}|^2 k_3^2}. \quad (3.19)$$

After this, employing the relations (3.9) and (3.11) we finally come to the conclusion that the sought modification of photon propagator in momentum space will have the same components as in (3.5) except for the cases $\mu = \nu = 1$ or 2 where the components will be

$$\begin{aligned} \tilde{D}_{xx}(k) &= \frac{1}{k^2 + i0} - \frac{k_2 \delta'(k_0)}{|\mathbf{k}|^2 k_3^2}, \\ \tilde{D}_{yy}(k) &= \frac{1}{k^2 + i0} + \frac{k_1 \delta'(k_0)}{|\mathbf{k}|^2 k_3^2}, \end{aligned} \quad (3.20)$$

and the generalized function $\delta'(k_0)$ acts according to the rule

$$\int \Phi(k_0) \delta'(k_0) dk_0 = -\Phi'(0).$$

4 Linear Confinement in QCD

4.1 Motivation

It is clear that considerations of the previous section are not confirmed experimentally in electrodynamics – there exist no elementary charged particles generating a constant magnetic field linear in r , the distance from particle, the given field obeying the Maxwell equations.

Another matter is quantum chromodynamics. The analogue of charge here is colour. The group $U(1)$ and the Maxwell equations are replaced by $SU(3)$ and the Yang-Mills equations but, as we have seen in Section 2, both the Maxwell equations and the Yang-Mills ones possess the confining solutions. Though quarks can unlikely be considered classical particles, after all, they accomplish a finite motion within a region with character size of order $1 \text{ fm} = 10^{-13} \text{ cm}$ and, as the explicit form of modulo square integrable solutions of the Dirac equation (5.4) in the confining $SU(3)$ -field (2.13)–(2.14) (relativistic bound states) shows (see Refs. [4, 5, 6, 35] and Section 7), the j -th colour component for the system of two quarks (e.g., for quarkonia) $\Psi_j \sim r^{\alpha_j} e^{-\beta_j r}$ with $\alpha_j > 0$, $\beta_j = \sqrt{\mu_0^2 - \omega_j^2 + g^2 b_j^2} > 0$, where $\omega = \sum \omega_j$ is an energetic level of system, $b_{1,2}$ are the parameters of linear interaction from the solutions (2.13)–(2.14), $b_3 = -(b_1 + b_2)$, r is a distance between quarks and Ψ_j is proved to be markedly different from zero only at $r \sim 1/\beta_j \sim 0.04 \text{ fm}$ (see Refs. [5, 6, 35] and Section 9), i.e., we deal with linear confinement of colour. Just the magnetic colour field defines the latter through the coefficients β_j . As a result, there are certain grounds to consider the qualitative physical picture from the previous section to occur just within QCD and the gluon propagator should be modified. The necessary modification can be realized in the same way as is done when deriving the standard gluon propagator (see, e.g., Ref. [7]),

i.e., through multiplying the propagator (3.5) [where the modification (3.20) is implied] by the factor δ^{ab} with $a, b = 1, \dots, 8$

$$\tilde{D}_{\mu\nu}^{ab}(k) = \delta^{ab} \tilde{D}_{\mu\nu}(k). \quad (4.1)$$

Under the circumstances the gluon propagator obtained will be able to lead to linear confinement at large distances (small momenta) while at small distances (large momenta) we can omit the additional addenda of order $|\mathbf{k}|^3$ in (3.20) and the resulting propagator will pass on to the standard gluon one used in the perturbative QCD.

It should be noted that during all the considerations we in fact dealt with the so-called Feynman gauge ($\alpha = 1$) but, as is not complicated to see, it is easy to generalize the results to an arbitrary α -gauge since this generalization concerns only the standard part of the propagator obtained and has been repeatedly discussed in literature (see, e.g., Ref. [7] and references therein).

Finally there are four important remarks.

Firstly, the fact is that the notion of propagator makes no sense for general nonlinear equations such as the Yang-Mills ones (2.12). If restricting, however, to the SU(3)-fields taking the values in the Cartan subalgebra of the SU(3)-Lee algebra (see Appendix B), i.e. in the subalgebra generated by the matrices λ_3 and λ_8 , then the equations (2.12) (at $J = 0$) become linear since the right-hand side of (2.12) is equal identically to zero for such field configurations (and for those gauge equivalent to the latter). The confining solutions (2.13)–(2.14) are just of the mentioned class. As a consequence, our modification of propagator holds true just in the latter set of SU(3)-Yang-Mills fields but the standard gluon propagator is tacitly supposed to correspond to the given class as well because it conforms to the Coulomb-like part of the solutions (2.13)–(2.14).

Secondly, one should say a few words concerning the nonrelativistic confining potentials often used, for example, in quarkonium theory (see, e.g., Ref. [14]). The confining potential between quarks here is usually modelled in the form $a/r + br$ with some constants a and b . It is clear, however, that from the QCD point of view the interaction between quarks should be described by the whole SU(3)-field $A_\mu = A_\mu^a T_a$, genuinely relativistic object, the nonrelativistic potential being only some component of A_t^a surviving in the nonrelativistic limit when the light velocity $c \rightarrow \infty$.

As has been mentioned in Refs. [5, 6] (see also Section 8), however, the U(1)- or SU(3)-field of form $A_t^a = Br^\gamma$, where B is a constant, may be solution of the Maxwell and Yang-Mills equations (2.3), (2.12) (at $J = 0$) only at $\gamma = -1$, i. e. in the Coulomb-like case.

As a result, the potentials employed in nonrelativistic approaches do not obey the Maxwell or Yang-Mills equations. The latter ones are essentially relativistic and, as we have seen, the components linear in r of the whole A_μ are different from A_t and related with (colour) magnetic field vanishing in the nonrelativistic limit. That is why the nonrelativistic confining potentials cannot be assumed as a basis when deriving the modification of gluon propagator under consideration.

Thirdly, it should be emphasized that the standard gluon propagator (as well as the corresponding photon one) is by itself essentially *nonperturbative* object [a fundamental solution (a Green function) of d'Alembert-like system (3.2)] that cannot be calculable by any perturbative techniques. Another matter that standard propagator is used in QCD *perturbation* theory for calculating quantum corrections, including for the propagator itself.

From mathematical point of view we should choose the Green function for the corresponding equations [SU(3)-Yang-Mills ones of (2.12)] which takes into account the necessary boundary conditions – linear confinement which is essentially *nonperturbative* phenomenon. The standard (nonperturbative) gluon propagator does not satisfy this condition. That is why we should modify the mentioned propagator in a *nonperturbative* way which can be done only on the basis of the corresponding exact *nonperturbative* solutions of the SU(3)-Yang-Mills equations. The resulting propagator in (4.1) is essentially *nonperturbative* one in each summand.

Fourthly, the structure of the obtained propagator of (3.20) shows that it has rather strong infrared singularities at $k \rightarrow 0$. Physical meaning of this is that quarks mainly emit and interchange the soft gluons (i.e., those with small k), so that gluon concentrations in the confining SU(3)-gluonic field are much greater than the estimates given in Section 9 (see also Ref. [35]) because the latter are the estimates for maximal possible gluon frequencies, i.e. for maximal possible gluon impulses (under the concrete situation of charmonium states). It is also clear that just magnetic part of the propagator (3.20) is responsible for larger portion of gluon concentrations since it has stronger infrared singularities than the electric part.

4.2 A Scenario for Linear Confinement

The above results and those of Refs. [4, 5, 6, 35] suggest the following mechanism of confinement to occur within the framework of QCD (at any rate, for mesons and quarkonia). The gluon exchange between quarks is realized by means of the propagator described above. At small distances one may neglect additional addenda of order $|k|^3$ in the propagator and we obtain the standard gluon one used in the perturbative QCD and, as a result, asymptotic freedom. At large distances the mentioned gluon exchange leads to the confining SU(3)-field of form (2.13)–(2.14) which may be considered classically (the gluon concentration becomes huge and gluons form the boson condensate – a classical field) and is a *nonperturbative* solution of the SU(3)-Yang-Mills equations. Under the circumstances mesons are the *relativistic bound states* described by the corresponding wave functions – *nonperturbative* modulo square integrable solutions of the Dirac equation in this confining SU(3)-field [4, 5, 6, 35]. For each meson there exists its own set of real constants a_j, A_j, b_j, B_j parametrizing the confining gluon field (2.13)–(2.14) (the mentioned gluon condensate) and the corresponding wave functions while the latter ones also depend on μ_0 , the reduced mass of the current masses of quarks forming meson [4, 5, 6, 35]. It is clear that constants $a_j, A_j, b_j, B_j, \mu_0$ should be extracted from experimental data and such a program has been just realized in Refs. [5, 6, 35] for quarkonia (see also Section 9).

Finally it should be emphasized that the reason why Nature chose the somewhat different propagator for gluons than the one for photons remains obscure within the framework of our considerations. To our mind, however, the questions of such a kind may be considered only from the cosmological positions, so that, for example, the problems concerning the number of quark flavours in QCD or a mechanism of breaking chiral symmetry in QCD can be scarcely resolved within the QCD framework in a single-valued way and a cosmological approach might be quite useful [15].

5 Confining Solutions of the $SU(N)$ -Yang-Mills Equations

5.1 Motivation

The previous section led us to the following problem: how to describe possible relativistic bound states in the confining $SU(3)$ -Yang-Mills fields? The sought description should be obviously based on the QCD-lagrangian. Let us write down this lagrangian (for one flavour and N quark colours) in arbitrary curvilinear (local) coordinates in Minkowski spacetime

$$\mathcal{L} = \bar{\Psi}\mathcal{D}\Psi - \mu_0\bar{\Psi}\Psi - \frac{1}{4}(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha})F_{\mu\nu}^a F_{\alpha\beta}^a, \mu < \nu, \alpha < \beta \quad (5.1)$$

where, if denoting $S(M)$ and ξ , respectively, the standard spinor bundle and N -dimensional vector one [equipped with a $SU(N)$ -connection with the corresponding connection and curvature matrices $A = A_\mu dx^\mu = A_\mu^a T_a dx^\mu$, $F = dA + gA \wedge A = F_{\mu\nu}^a T_a dx^\mu \wedge dx^\nu$, see Appendix B] over Minkowski spacetime, we can construct tensorial product $\Xi = S(M) \otimes \xi$. It is clear that Ψ is just a section of the latter bundle, i. e. Ψ can be chosen in the form $\Psi = (\Psi_1, \dots, \Psi_N)$ with the four-dimensional Dirac spinors ψ_j representing the j -th colour component while $\bar{\Psi} = \Psi^\dagger(\gamma^0 \otimes I_N)$ is the adjoint spinor, (\dagger) stands for hermitian conjugation, I_N is the unit matrix $N \times N$, \otimes means tensorial product of matrices, μ_0 is a mass parameter, \mathcal{D} is the Dirac operator with coefficients in ξ (see below). At last, we take the condition $\text{Tr}(T_a T_b) = K\delta_{ab}$ with some real K so that the third addendum in (5.1) has the form $G(F, F)/(4K)$ (see Appendix A), where coefficient $1/(4K)$ is chosen from physical considerations.

Of course, the most physically important case is that of group $SU(3)$, i.e., three colors of quarks, however, it makes sense to have analysed the general $SU(N)$ -case with arbitrary N as well because during a long time there is a firm belief in that considering the limit $N \rightarrow \infty$ to be rather (or even extremely) important for understanding of the real 4D QCD with three colors. The spectrum of speculations on this topic ranges from phenomenological and lattice approaches (for more details see, e. g., Refs. [16] and references therein) to the exotic scenarios based on M -theory and strings (see, e. g., Ref. [17] and references therein).

From general considerations the explicit form of the operator \mathcal{D} in local coordinates x^μ on Minkowski spacetime can be written as follows

$$\mathcal{D} = i(\gamma^e \otimes I_N)E_e^\mu \left(\partial_\mu \otimes I_N - \frac{1}{2}\omega_{\mu ab}\gamma^a\gamma^b \otimes I_N - igA_\mu \right), a < b, \quad (5.2)$$

where g is a gauge coupling constant, the forms $\omega_{ab} = \omega_{\mu ab} dx^\mu$ obey the Cartan structure equations $de^a = \omega_b^a \wedge e^b$ with exterior derivative d , while the orthonormal basis $e^a = e_\mu^a dx^\mu$ in cotangent bundle and dual basis $E_a = E_a^\mu \partial_\mu$ in tangent bundle are connected by the relations $e^a(E_b) = \delta_b^a$. At last, matrices γ^a represent the Clifford algebra of the corresponding quadratic form in \mathbb{C}^4 . Below we shall deal only with 4D lorentzian case (quadratic form $Q_{1,3} = x_0^2 - x_1^2 - x_2^2 - x_3^2$). For this we take the following choice for γ^a

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma^b = \begin{pmatrix} 0 & \sigma_b \\ -\sigma_b & 0 \end{pmatrix}, b = 1, 2, 3, \quad (5.3)$$

where σ_b denote the ordinary Pauli matrices (see Appendix B). It should be noted that, in lorentzian case, Greek indices μ, ν, \dots are raised and lowered with $g_{\mu\nu}$ of (1.1) or its inverse $g^{\mu\nu}$ and Latin indices a, b, \dots are raised and lowered by $\eta_{ab} = \eta^{ab} = \text{diag}(1, -1, -1, -1)$ except for Latin indices connected with $\text{SU}(N)$ -Lie algebras, so that $e_\mu^a e_\nu^b g^{\mu\nu} = \eta^{ab}$, $E_a^\mu E_b^\nu g_{\mu\nu} = \eta_{ab}$ and so on but $T_a = T^a$.

Under the circumstances we can obtain the following equations according to the standard prescription of Lagrange approach from (5.1)

$$\mathcal{D}\Psi = \mu_0 \Psi, \quad (5.4)$$

$$d * F = g(*F \wedge A - A \wedge *F) + gJ, \quad (5.5)$$

where $*$ means the Hodge star operator conforming to a Minkowski metric, for instance, in the form of (1.1), while the source J (a nonabelian $\text{SU}(N)$ -current) is

$$J = j_\mu^a T_a * (dx^\mu) = *j = *(j_\mu^a T_a dx^\mu) = *(j^a T_a) \quad (5.6)$$

where currents

$$j^a = j_\mu^a dx^\mu = \bar{\Psi}(\gamma_\mu \otimes I_N) T^a \Psi dx^\mu,$$

so summing over $a = 1, \dots, N^2 - 1$ is implied in (5.1) and (5.6).

When using the relation (see, e. g. Refs. [22])

$\gamma^c E_c^\mu \omega_{\mu ab} \gamma^a \gamma^b = \omega_{\mu ab} \gamma^\mu \gamma^a \gamma^b = -\text{div}(\gamma)$ with matrix 1-form $\gamma = \gamma_\mu dx^\mu$ [where div is defined by relation (B.4)] and also the fact that $(\gamma^\mu)^\dagger \gamma^0 = \gamma^0 \gamma^\mu$, the Dirac equation for spinor $\bar{\Psi}$ will be

$$i\partial_\mu \bar{\Psi}(\gamma^\mu \otimes I_N) + \frac{i}{2} \bar{\Psi} \text{div}(\gamma) \otimes I_N - g \bar{\Psi}(\gamma^\mu \otimes I_N) A_\mu^a T_a = -\mu_0 \bar{\Psi}. \quad (5.4')$$

Then multiplying (5.4) by $\bar{\Psi} T_a$ from left and (5.4') by $T_a \Psi$ from right and adding the obtained equations, we get $\text{div}(j^a) = \text{div}(j) = 0$ if spinor Ψ obeys the Dirac equation (5.4).

The question now is how to connect the sought relativistic bound states with the system (5.4)–(5.5). To understand it let us apply to the experience related with QED. In the latter case lagrangian looks like (5.1) with changing group $\text{SU}(N) \rightarrow \text{U}(1)$ so Ψ will be just a four-dimensional Dirac spinor. Then, as is known (see, e. g. Ref. [23]), when passing

on to the nonrelativistic limit the Dirac equation (5.4) converts into the Pauli equation and further, if neglecting the particle spin, into the Schrödinger equation, parameter μ_0 becoming the reduced mass of two-body system. The modulo square integrable solutions of the Schrödinger equation just describe bound states of a particle with mass μ_0 , or, that is equivalent, of the corresponding two-body system. Historically, however, everything was just vice versa. At first there appeared the Schrödinger equation, then the Pauli and Dirac ones and only then the QED lagrangian. In its turn, possibility of writing two-body Schrödinger equation on the whole owed to the fact that the corresponding two-body problem in classical nonrelativistic (newtonian) mechanics was well posed and actually quantizing the latter gave two-body Schrödinger equation. Another matter was Dirac equation. Up to now nobody can say what two-body problem in classical relativistic (einsteinian) mechanics could correspond to Dirac equation. The fact is that the two-body problem in classical relativistic mechanics has so far no single-valued statement. Conventionally, therefore, Dirac equation in QED is treated as the relativistic wave equation describing one particle with spin one half in an external electromagnetic field.

There is, however, one important exclusion – the hydrogen atom. When solving the Dirac equation here one considers mass parameter μ_0 to be equal to the electron mass and one gets the so-called Sommerfeld formula for hydrogen atom levels which passes on to the standard Schrödinger formula for hydrogen atom spectrum in nonrelativistic limit (for more details see, e. g. Ref. [23] and also Subsection 6.2). But in the Schrödinger formula mass parameter μ_0 is equal to the reduced mass of electron and proton. As a consequence, it is tacitly supposed that in Dirac equation the mass parameter should be equal to the same reduced mass of electron and proton as in Schrödinger equation. Just the mentioned reduced mass is approximately equal to that of electron but, exactly speaking, it is not the case. We remind that for the problem under discussion (hydrogen atom) the external field is the Coulomb electric one between electron and proton, essentially nonrelativistic object in the sense that it does not vanish in nonrelativistic limit at $c \rightarrow \infty$. If now to place hydrogen atom in a magnetic field then obviously spectrum of bound states will also depend on parameters describing the magnetic field. The latter, however, is essentially relativistic object and vanishes at $c \rightarrow \infty$ because, as is well known, in the world with $c = \infty$ there exist no magnetic fields (see any elementary textbook on physics, e. g. Ref. [24]). But it is clear that spectrum should as before depend of μ_0 as well and we can see that μ_0 is the same reduced mass as before since in nonrelativistic limit we again should come to the hydrogen atom spectrum with the reduced mass. So we can draw the conclusion that if an electromagnetic field is a combination of electric Coulomb field between two charged particles and some magnetic field (which may be generated by the particles themselves) then there are certain grounds to consider the given (quantum) two-body problem to be equivalent to the one of motion for one particle with usual reduced mass in the mentioned electromagnetic field. As a result, we can use the Dirac equation for finding possible relativistic bound states for such a particle implying that this is really some description of the corresponding two-body problem.

Actually in QED the situation is just as the described one but magnetic field is usually

weak and one may restrict oneself to some corrections from this field to the nonrelativistic Coulomb spectrum (e.g., in the Zeeman effect). If the magnetic field is strong then one should solve just Dirac equation in a nonperturbative way (see, e.g. Ref. [25]). The latter situation seems to be natural in QCD where the corresponding magnetic (colour) field should be very strong because just it provides linear confinement of quarks as we shall see below (see also Section 9).

At last, we should make an important point that in QED the mentioned electromagnetic field is by definition always a solution of the Maxwell equations so within QCD we should require the confining SU(3)-field to be a solution of Yang-Mills equations. Consequently, returning to the system (5.4)–(5.5), we can suggest to describe relativistic bound states of two quarks (mesons) in QCD by the compatible solutions of the given system. To be more precise, the meson wave functions should be the *nonperturbative* modulo square integrable solutions of Dirac equation (5.4) (with the above reduced mass μ_0) in the confining SU(3)-Yang-Mills field being a *nonperturbative* solution of (5.5). In general case, however, the analysis of (5.5) is difficult because of availability of the nonabelian current J of (5.6) in the right-hand side of (5.5) but we may use the circumstance that the corresponding modulo square integrable solutions of Dirac equation (5.4) might consist from the components of form $\Psi_j \sim r^{\alpha_j} e^{-\beta_j r}$ with some $\alpha_j > 0$, $\beta_j > 0$ which entails all the components of the current J to be modulo $\ll 1$ at each point of Minkowski space. The latter will allow us to put $J \approx 0$ and we shall come to the problem of finding the confining solutions for the Yang-Mills equations of (5.5) with $J = 0$ (according to Section 1 such solutions are supposed to be spherically symmetric and to contain only the components of the SU(N)-field which are Coulomb-like or linear in r) and after inserting the found solutions into Dirac equation (5.4) we should require the corresponding solutions of Dirac equation to have the above necessary behaviour. Under the circumstances the problem becomes self-consistent and can be analyzable.

It is clear that all the above considerations can be justified only by comparison with experimental data but now we obtained some intelligible programme of further activity. So let us pass on to its realization.

5.2 Role of Diagonal Gauge

Let us in detail write out $A_\mu = A_\mu^a T_a$ of the Dirac operator from (5.2) employing the SU(N)-Lie algebra realizations from Appendix B. We obtain at $N = 2, 3, 4$ respectively

$$A_\mu^a \sigma_a = \begin{pmatrix} A_\mu^3 & A_\mu^1 - iA_\mu^2 \\ A_\mu^1 + iA_\mu^2 & -A_\mu^3 \end{pmatrix},$$

$$A_\mu^a \lambda_a = \begin{pmatrix} A_\mu^3 + \frac{1}{\sqrt{3}}A_\mu^8 & A_\mu^1 - iA_\mu^2 & A_\mu^4 - iA_\mu^5 \\ A_\mu^1 + iA_\mu^2 & -A_\mu^3 + \frac{1}{\sqrt{3}}A_\mu^8 & A_\mu^6 - iA_\mu^7 \\ A_\mu^4 + iA_\mu^5 & A_\mu^6 + iA_\mu^7 & -\frac{2}{\sqrt{3}}A_\mu^8 \end{pmatrix},$$

$$A_\mu^a T_a =$$

$$\begin{pmatrix} A_\mu^3 + A_\mu^6 + A_\mu^{15} & z_1 & z_2 & z_3 \\ z_1^* & -A_\mu^3 + A_\mu^6 - A_\mu^{15} & z_4 & z_5 \\ z_2^* & z_4^* & A_\mu^3 - A_\mu^6 - A_\mu^{15} & z_6 \\ z_3^* & z_5^* & z_6^* & -A_\mu^3 - A_\mu^6 + A_\mu^{15} \end{pmatrix} \quad (5.7)$$

with $z_1 = A_\mu^1 + A_\mu^9 - i(A_\mu^2 + A_\mu^{12})$, $z_2 = A_\mu^4 + A_\mu^{13} - i(A_\mu^5 + A_\mu^{14})$, $z_3 = A_\mu^7 - A_\mu^{11} - i(A_\mu^8 + A_\mu^{10})$, $z_4 = A_\mu^7 + A_\mu^{11} - i(A_\mu^8 - A_\mu^{10})$, $z_5 = A_\mu^4 - A_\mu^{13} - i(A_\mu^5 - A_\mu^{14})$, $z_6 = A_\mu^1 - A_\mu^9 - i(A_\mu^2 - A_\mu^{12})$, where (*) signifies complex conjugation. Then it is natural to put $A_\mu^a = 0$ with $a = 1, 2$ at $N = 2$, $A_\mu^a = 0$ with $a = 1, 2, 4, 5, 6, 7$ at $N = 3$, $A_\mu^a = 0$ with $a = 1, 2, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14$ at $N = 4$ so long as the Dirac equation (5.4) in such a gauge takes the simplest form. We further call this gauge *diagonal* one. Really Dirac equation (5.4) in diagonal gauge splits into the system of Dirac equations for components Ψ_j . Namely, at $N = 2$ the system is

$$\begin{aligned} i\gamma^e E_e^\mu \left[\partial_\mu - \frac{1}{2} \omega_{\mu ab} \gamma^a \gamma^b - ig A_\mu^3 \right] \Psi_1 &= \mu_0 \Psi_1, \\ i\gamma^e E_e^\mu \left[\partial_\mu - \frac{1}{2} \omega_{\mu ab} \gamma^a \gamma^b + ig A_\mu^3 \right] \Psi_2 &= \mu_0 \Psi_2, \end{aligned} \quad (5.8)$$

while at $N = 3$ it is

$$\begin{aligned} i\gamma^e E_e^\mu \left[\partial_\mu - \frac{1}{2} \omega_{\mu ab} \gamma^a \gamma^b - ig \left(A_\mu^3 + \frac{1}{\sqrt{3}} A_\mu^8 \right) \right] \Psi_1 &= \mu_0 \Psi_1, \\ i\gamma^e E_e^\mu \left[\partial_\mu - \frac{1}{2} \omega_{\mu ab} \gamma^a \gamma^b - ig \left(-A_\mu^3 + \frac{1}{\sqrt{3}} A_\mu^8 \right) \right] \Psi_2 &= \mu_0 \Psi_2, \\ i\gamma^e E_e^\mu \left[\partial_\mu - \frac{1}{2} \omega_{\mu ab} \gamma^a \gamma^b - ig \left(-\frac{2}{\sqrt{3}} A_\mu^8 \right) \right] \Psi_3 &= \mu_0 \Psi_3, \end{aligned} \quad (5.9)$$

and, at last, at $N = 4$ the corresponding system is

$$\begin{aligned} i\gamma^e E_e^\mu \left[\partial_\mu - \frac{1}{2} \omega_{\mu ab} \gamma^a \gamma^b - ig \left(A_\mu^3 + A_\mu^6 + A_\mu^{15} \right) \right] \Psi_1 &= \mu_0 \Psi_1, \\ i\gamma^e E_e^\mu \left[\partial_\mu - \frac{1}{2} \omega_{\mu ab} \gamma^a \gamma^b - ig \left(-A_\mu^3 + A_\mu^6 - A_\mu^{15} \right) \right] \Psi_2 &= \mu_0 \Psi_2, \\ i\gamma^e E_e^\mu \left[\partial_\mu - \frac{1}{2} \omega_{\mu ab} \gamma^a \gamma^b - ig \left(A_\mu^3 - A_\mu^6 - A_\mu^{15} \right) \right] \Psi_3 &= \mu_0 \Psi_3, \\ i\gamma^e E_e^\mu \left[\partial_\mu - \frac{1}{2} \omega_{\mu ab} \gamma^a \gamma^b - ig \left(-A_\mu^3 - A_\mu^6 + A_\mu^{15} \right) \right] \Psi_4 &= \mu_0 \Psi_4. \end{aligned} \quad (5.10)$$

It is clear that in diagonal gauge the $SU(N)$ -Yang-Mills fields are described by matrices $A_\mu = A_\mu^a T_a$ taking their values in the Cartan subalgebra of the conforming $SU(N)$ -Lie algebra (see Appendix B).

Due to that $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$, we have $A \wedge A = A_\mu^a A_\nu^b [T_a, T_b] dx^\mu \wedge dx^\nu$, $\mu < \nu$. Consequently for the $SU(N)$ -Yang-Mills with values in the Cartan subalgebra

$A \wedge A = 0$ since commutator $[T_a, T_b]$ for diagonal matrices is always equal to zero and the Cartan subalgebras of $SU(N)$ -groups just consist from diagonal matrices (see Appendix B). Accordingly the curvature matrix (field strength) $F = dA + gA \wedge A = dA$ while the right-hand side of the Yang-Mills equations (5.5) (at $J = 0$) is identically equal to zero since matrix $*F$ is also diagonal and then $*F \wedge A = A \wedge *F$. This fact strongly simplifies the task of searching for confining solutions because the equations (5.5) convert into

$$d * F = 0. \quad (5.11)$$

It should be emphasized that such a simplification is dictated by the wish to obtain the simplest form for the Dirac equation (5.4). Clearly, all the results obtained in diagonal gauge will hold true in any gauge connected with the diagonal one by some gauge transformation due to the fact of gauge invariance of the Yang-Mills equations (5.5). It turns out, however, that the confining solutions obtained in this way has the property of uniqueness in a certain sense which will be discussed in Section 8 and now let us pass on to finding confining solutions.

5.3 U(1)-case

We have already discussed this case in Section 2, where the corresponding confining solution was found in the form (2.6). But it could seem that when searching for the solution the ansatz used was not the most general one. Really, we took the ansatz in the form $A = A_t(r)dt + A_\varphi(r)d\varphi$. It seems that the most general form is $A = A_t(r)dt + A_r(r)dr + A_\vartheta(r)d\vartheta + A_\varphi(r)d\varphi$. Let us discuss it in more details.

For the latter ansatz we have $F = dA = -\partial_r A_t dt \wedge dr + \partial_r A_\vartheta dr \wedge d\vartheta + \partial_r A_\varphi dr \wedge d\varphi$ for an arbitrary $A_r(r)$. Then, according to (A.6), we obtain

$$*F = (r^2 \sin \vartheta) \partial_r A_t d\vartheta \wedge d\varphi + \sin \vartheta \partial_r A_\vartheta dt \wedge d\varphi - \frac{1}{\sin \vartheta} \partial_r A_\varphi dt \wedge d\vartheta \quad (5.12)$$

which entails

$$\begin{aligned} d * F = & \sin \vartheta \partial_r (r^2 \partial_r A_t) dr \wedge d\vartheta \wedge d\varphi - \sin \vartheta \partial_r^2 A_\vartheta dt \wedge dr \wedge d\varphi - \\ & \cos \vartheta \partial_r A_\vartheta dt \wedge d\vartheta \wedge d\varphi + \frac{1}{\sin \vartheta} \partial_r^2 A_\varphi dt \wedge dr \wedge d\vartheta = 0, \end{aligned} \quad (5.13)$$

wherefrom one can conclude that

$$\partial_r (r^2 \partial_r A_t) = 0, \quad \partial_r^2 A_\varphi = 0, \quad (5.14)$$

$$\partial_r^2 A_\vartheta = \partial_r A_\vartheta = 0. \quad (5.15)$$

This yields the solutions (2.6) while we draw the conclusion that $A_\vartheta = C_1$ with some constant C_1 . But then the Lorentz condition (2.2) for the given ansatz entails

$$\sin \vartheta \partial_r (r^2 A_r) + \partial_\vartheta (\sin \vartheta A_\vartheta) = 0,$$

or

$$\partial_r(r^2 A_r) + \cot \vartheta A_\vartheta = 0, \quad (5.16)$$

which entails $A_r = -A_\vartheta \cot \vartheta / r + C_2 / r^2$ with a constant C_2 . But the confining solutions, as we accept in the given paper, should be spherically symmetric and contain only the components which are Coulomb-like or linear in r , so one should put $C_1 = C_2 = 0$. Consequently, the ansatz $A = A_t(r)dt + A_\varphi(r)d\varphi$ is most general and we can consider $A_r = A_\vartheta = 0$ without loss of generality.

Let us describe one class of the confining nonspherically symmetric solutions of the Maxwell equations (2.3) (at $J = 0$) that can be obtained with the aid of the ansatz $A = A_t(r)dt + A_\varphi(r, \vartheta)d\varphi$, i. e. now we consider the component A_φ depending also on ϑ . It is evident that the Lorentz condition (2.2) is automatically fulfilled for the given ansatz. We shall have $F = dA = -\partial_r A_t dt \wedge dr + \partial_r A_\varphi dr \wedge d\varphi + \partial_\vartheta A_\varphi d\vartheta \wedge d\varphi$ and $*F = r^2 \sin \vartheta \partial_r A_t d\vartheta \wedge d\varphi - \frac{1}{\sin \vartheta} \partial_r A_\varphi dt \wedge d\vartheta + \frac{1}{r^2 \sin \vartheta} \partial_\vartheta A_\varphi dt \wedge dr$. Then Eq. (2.3) (at $J = 0$) entails

$$\partial_r(r^2 \partial_r A_t) = 0, \quad (5.17)$$

$$r^2 \partial_r^2 A_\varphi + \sin \vartheta \partial_\vartheta \left(\frac{1}{\sin \vartheta} \partial_\vartheta A_\varphi \right) = 0. \quad (5.18)$$

We shall not here discuss the general form of the solution for Eq. (5.18) and only write out the possible solution of (5.17)–(5.18) which is useful to us in the present paper in the form slightly modifying (2.6)

$$A_t = \frac{a}{r} + A, \quad A_\varphi = br + B - K \cos \vartheta \quad (5.19)$$

with some constants a, b, A, B, K parametrizing solution.

5.4 $N = 2$

Remarks done in previous subsection about the Lorentz condition will hold true for any group $SU(N)$ (see Subsection 8.1), so at $N = 2$ we put $A_{r,\vartheta}^3 = 0$. After this we search for the solution of (B.3) (at $J = 0$) in the form $A = A_t(r)dt + A_\varphi(r)d\varphi$ with $A_{t,\varphi} = A_{t,\varphi}^3 \sigma_3$. Along the above lines it is then easy to come to the system

$$\partial_r(r^2 \partial_r A_t) = 0, \quad \partial_r^2 A_\varphi = 0, \quad (5.20)$$

and we write down the solutions of (5.20) needed to insert into (5.8)

$$A_t^3 = -\frac{a}{r} + A, \quad (5.21)$$

$$A_\varphi^3 = br + B \quad (5.22)$$

with some constants a, A, b, B parametrizing solutions.

Class of the confining nonspherically symmetric solutions of (B.3) can be obtained with the aid of the ansatz $A = A_t(r)dt + A_\varphi(r, \vartheta)d\varphi$. We shall have

$$\partial_r(r^2\partial_r A_t) = 0, \quad (5.23)$$

$$r^2\partial_r^2 A_\varphi + \sin\vartheta\partial_\vartheta\left(\frac{1}{\sin\vartheta}\partial_\vartheta A_\varphi\right) = 0. \quad (5.24)$$

It is clear that Eq. (5.23) gives the same solution of (5.21) while one possible solution of (5.24) useful to us in the present paper is

$$A_\varphi^3 = -K \cos\vartheta + br + B \quad (5.25)$$

with some real constants K, b, B parametrizing solution.

5.5 $N = 3$

In the given case we put $A_{r,\vartheta}^{3,8} = 0$. After this the ansatz in the form $A = A_t(r)dt + A_\varphi(r)d\varphi$ with $A_{t,\varphi} = A_{t,\varphi}^3\lambda_3 + A_{t,\varphi}^8\lambda_8$ yields (at $J = 0$) the solutions (2.13). The corresponding class of the confining nonspherically symmetric solutions of (B.3) (at $J = 0$) can be obtained with the aid of the ansatz $A = A_t(r)dt + A_\varphi(r, \vartheta)d\varphi$ which gives the same component A_t as in (2.13) while

$$\begin{aligned} A_\varphi^3 + \frac{1}{\sqrt{3}}A_\varphi^8 &= -K_1 \cos\vartheta + b_1 r + B_1, \\ -A_\varphi^3 + \frac{1}{\sqrt{3}}A_\varphi^8 &= -K_2 \cos\vartheta + b_2 r + B_2, \\ -\frac{2}{\sqrt{3}}A_\varphi^8 &= (K_1 + K_2) \cos\vartheta - (b_1 + b_2)r - (B_1 + B_2) \end{aligned} \quad (5.26)$$

with some real constants K_j, b_j, B_j parametrizing solution.

5.6 $N = 4$

We put $A_{r,\vartheta}^{3,6,15} = 0$ and at $J = 0$ the ansatz $A = A_t(r)dt + A_\varphi(r)d\varphi$ with $A_{t,\varphi} = A_{t,\varphi}^3T_3 + A_{t,\varphi}^6T_8 + A_{t,\varphi}^{15}T_{15}$ gives rise to the solutions of (B.3) in the form

$$\begin{aligned} A_t^3 + A_t^6 + A_t^{15} &= -\frac{a_1}{r} + A_1, \quad -A_t^3 + A_t^6 - A_t^{15} = -\frac{a_2}{r} + A_2, \\ A_t^3 - A_t^6 - A_t^{15} &= -\frac{a_3}{r} + A_3, \quad -A_t^3 - A_t^6 + A_t^{15} = \frac{a_1 + a_2 + a_3}{r} - (A_1 + A_2 + A_3), \\ A_\varphi^3 + A_\varphi^6 + A_\varphi^{15} &= b_1 r + B_1, \quad -A_\varphi^3 + A_\varphi^6 - A_\varphi^{15} = b_2 r + B_2, \\ A_\varphi^3 - A_\varphi^6 - A_\varphi^{15} &= b_3 r + B_3, \quad -A_\varphi^3 - A_\varphi^6 + A_\varphi^{15} = -(b_1 + b_2 + b_3)r - (B_1 + B_2 + B_3), \end{aligned} \quad (5.27)$$

where real constants a_j, A_j, b_j, B_j parametrize the solutions. The corresponding class of the confining nonspherically symmetric solutions of (B.3) (at $J = 0$) can be obtained with

the aid of the ansatz $A = A_t(r)dt + A_\varphi(r, \vartheta)d\varphi$ which gives the same component A_t as in (5.27) while

$$\begin{aligned} A_\varphi^3 + A_\varphi^6 + A_\varphi^{15} &= b_1 r + B_1 - K_1 \cos \vartheta, \quad -A_\varphi^3 + A_\varphi^6 - A_\varphi^{15} = b_2 r + B_2 - K_2 \cos \vartheta, \\ A_\varphi^3 - A_\varphi^6 - A_\varphi^{15} &= b_3 r + B_3 - K_3 \cos \vartheta, \\ -A_\varphi^3 - A_\varphi^6 + A_\varphi^{15} &= -(b_1 + b_2 + b_3)r - (B_1 + B_2 + B_3) + (K_1 + K_2 + K_3) \cos \vartheta \end{aligned} \quad (5.28)$$

with real constants K_j, b_j, B_j .

6 Spectrum of Bound States in the Coulomb-Like Case

The question now is how to find the modulo square integrable solutions of Dirac equation (5.4) when inserting the confining solutions described in previous Section into it. We shall need some results about spectrum of the euclidean Dirac operator on the unit two-dimensional sphere \mathbb{S}^2 in the form obtained in Refs. [26].

6.1 Results from the Black Hole Theory about Eigenspinors of the (Twisted) Euclidean Dirac Operator on \mathbb{S}^2

When separating variables in (5.4) (see next Subsection) there naturally arises the euclidean Dirac operator \mathcal{D}_0 on the unit two-dimensional sphere \mathbb{S}^2 and we should know its eigenvalues with the corresponding eigenspinors. Such a problem also arises in the black hole theory while describing the so-called twisted spinors on Schwarzschild and Reissner-Nordström black holes and it was analysed in Refs. [26], so we can use the results obtained therein for our aims. Let us adduce the necessary relations.

Let us consider $2k$ -dimensional (pseudo)riemannian manifold M for which $H^1(M, \mathbb{Z}_2)$, the first cohomology group with coefficients in \mathbb{Z}_2 , is equal to zero while $H^2(M, \mathbb{Z})$, the second cohomology group with coefficients in \mathbb{Z} , is equal to \mathbb{Z} . Then standard topological results [18, 19, 20] say to us that over M there exists the only so-called Spin-structure whereas there is countable number of complex line bundles over M . As a consequence, each complex line bundle can be characterized by an integer n which in what follows will be called its Chern number. Under this situation, if denoting $S(M)$ the only standard spinor bundle over M and ξ the complex line bundle with Chern number n , we can construct tensorial product $S(M) \otimes \xi$. Under the circumstances we obtain the *twisted Dirac operator* $\mathcal{D}_n : S(M) \otimes \xi \rightarrow S(M) \otimes \xi$, so the eigenvalue equation for corresponding spinors Φ as sections of the bundle $S(M) \otimes \xi$ may look as follows

$$\mathcal{D}_n \Phi = \lambda \Phi, \quad (6.1)$$

and we can call (standard) spinors corresponding to $n = 0$ (trivial complex line bundle ξ) *untwisted* while the rest of the spinors with $n \neq 0$ should be referred to as *twisted*.

From general considerations [18, 19, 20] the explicit form of the operator \mathcal{D}_n in local coordinates x^μ on a $2k$ -dimensional (pseudo)riemannian manifold can be written as follows

$$\mathcal{D}_n = i\gamma^\mu \nabla_\mu \equiv i\gamma^c E_c^\mu (\partial_\mu - \frac{1}{2}\omega_{\mu ab}\gamma^a\gamma^b - ieA_\mu), \quad a < b, \quad (6.2)$$

where $A = A_\mu dx^\mu$ is a connection in the bundle ξ and the forms $\omega_{ab} = \omega_{\mu ab} dx^\mu$ obey the Cartan structure equations $de^a = \omega_b^a \wedge e^b$ with exterior derivative d , while the orthonormal basis $e^a = e_\mu^a dx^\mu$ in cotangent bundle and dual basis $E_a = E_a^\mu \partial_\mu$ in tangent bundle are connected by the relations $e^a(E_b) = \delta_b^a$. At last, matrices γ^a represent the Clifford algebra of the corresponding quadratic form in \mathbb{C}^{2k} . Below we shall deal only with 2D euclidean case of the unit sphere \mathbb{S}^2 ($k = 1$, quadratic form $Q_2 = x_0^2 + x_1^2$).

As for the connection A_μ in bundle ξ then the suitable one can be found, for example, in Refs. [8] and is

$$A = A_\mu dx^\mu = -\frac{n}{e} \cos \vartheta d\varphi. \quad (6.3)$$

Under the circumstances, as was shown in Refs. [8], integrating $F = dA$ over the unit sphere \mathbb{S}^2 gives rise to the Dirac charge quantization condition

$$\int_{\mathbb{S}^2} F = 4\pi \frac{n}{e} = 4\pi q \quad (6.4)$$

with magnetic charge q , so we can identify the coupling constant e with electric charge.

As was discussed in Refs. [26], the natural form of \mathcal{D}_n in local coordinates ϑ, φ on the unit sphere \mathbb{S}^2 looks as follows

$$\begin{aligned} \mathcal{D}_n = & -i\sigma_1 \left[i\sigma_2 \partial_\vartheta + i\sigma_3 \frac{1}{\sin \vartheta} \left(\partial_\varphi - \frac{1}{2}\sigma_2 \sigma_3 \cos \vartheta + in \cos \vartheta \right) \right] = \\ & \sigma_1 \sigma_2 \partial_\vartheta + \frac{1}{\sin \vartheta} \sigma_1 \sigma_3 \partial_\varphi - \frac{\cot \vartheta}{2} \sigma_1 \sigma_2 + in \sigma_1 \sigma_3 \cot \vartheta \end{aligned} \quad (6.5)$$

with the Pauli matrix σ_j (see Appendix B), so that $\sigma_1 \mathcal{D}_n = -\mathcal{D}_n \sigma_1$. As is not complicated to see, the operator \mathcal{D}_n has the form (6.2) with $\gamma^0 = -i\sigma_1 \sigma_2$, $\gamma^1 = -i\sigma_1 \sigma_3$, $e^0 = d\vartheta$, $e^1 = \sin \vartheta d\varphi$, $E_0 = \partial_\vartheta$, $E_1 = \partial_\varphi / \sin \vartheta$, $\omega_{01} = \cos \vartheta d\varphi$, $A_\mu dx^\mu = -\frac{n}{e} \cos \vartheta d\varphi$.

The equation (6.1) was explored in Refs. [26]. Spectrum of \mathcal{D}_n consists of the numbers $\lambda = \pm \sqrt{(l+1)^2 - n^2}$ with multiplicity $2(l+1)$ of each one, where $l = 0, 1, 2, \dots, l \geq |n|$. Let us introduce the number m such that $-l \leq m \leq l+1$ and the corresponding number $m' = m - 1/2$ so $|m'| \leq l + 1/2$. Then the conforming eigenspinors of operator \mathcal{D}_n are

$$\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \Phi_{\mp \lambda} = \frac{C}{2} \begin{pmatrix} P_{m'n-1/2}^k \pm P_{m'n+1/2}^k \\ P_{m'n-1/2}^k \mp P_{m'n+1/2}^k \end{pmatrix} e^{-im'\varphi} \quad (6.6)$$

with the coefficient $C = \sqrt{\frac{l+1}{2\pi}}$. These spinors form an orthonormal basis in $L_2^2(\mathbb{S}^2)$ for each n and are subject to the normalization condition

$$\int_{\mathbb{S}^2} \Phi^\dagger \Phi d\Omega = \int_0^\pi \int_0^{2\pi} (|\Phi_1|^2 + |\Phi_2|^2) \sin \vartheta d\vartheta d\varphi = 1, \quad (6.7)$$

where (\dagger) stands for hermitian conjugation. As to functions $P_{m'n'}^k(\cos \vartheta)$ then they can be chosen by miscellaneous ways, for instance, as follows (see, e. g., Ref. [27])

$$P_{m'n'}^k(\cos \vartheta) = i^{-m'-n'} \sqrt{\frac{(k-m')!(k-n')!}{(k+m')!(k+n')!}} \left(\frac{1+\cos \vartheta}{1-\cos \vartheta} \right)^{\frac{m'+n'}{2}} \times \\ \times \sum_{j=\max(m',n')}^k \frac{(k+j)!i^{2j}}{(k-j)!(j-m')!(j-n')!} \left(\frac{1-\cos \vartheta}{2} \right)^j \quad (6.8)$$

with the orthogonality relation at m', n' fixed

$$\int_0^\pi P_{m'n'}^{*k}(\cos \vartheta) P_{m'n'}^k(\cos \vartheta) \sin \vartheta d\vartheta = \frac{2}{2k+1} \delta_{kk'}, \quad (6.9)$$

where $(*)$ signifies complex conjugation. It should be noted that square of \mathcal{D}_n is

$$\mathcal{D}_n^2 = \mathcal{D}_0^2 - \frac{2in \cos \vartheta \partial_\varphi - n^2}{\sin^2 \vartheta} + in \frac{1}{\sin^2 \vartheta} \sigma_2 \sigma_3 - n^2 \quad (6.10)$$

with

$$\mathcal{D}_0^2 = -\Delta_{\mathbb{S}^2} + \sigma_2 \sigma_3 \frac{\cos \vartheta}{\sin^2 \vartheta} \partial_\varphi + \frac{1}{4 \sin^2 \vartheta} + \frac{1}{4}, \quad (6.11)$$

while laplacian on the unit sphere is

$$\Delta_{\mathbb{S}^2} = \frac{1}{\sin \vartheta} \partial_\vartheta \sin \vartheta \partial_\vartheta + \frac{1}{\sin^2 \vartheta} \partial_\varphi^2 = \partial_\vartheta^2 + \cot \vartheta \partial_\vartheta + \frac{1}{\sin^2 \vartheta} \partial_\varphi^2, \quad (6.12)$$

so the relation (6.10) is a particular case of the so-called Weitzenböck-Lichnerowicz formulas (see Refs. [18, 19, 20]). Then from (6.1) it follows $\mathcal{D}_n^2 \Phi = \lambda^2 \Phi$ and, when using the ansatz $\Phi = P(\vartheta) e^{-im'\varphi} = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} e^{-im'\varphi}$, $P_{1,2} = P_{1,2}(\vartheta)$, the equation $\mathcal{D}_n^2 \Phi = \lambda^2 \Phi$ turns into

$$\left(-\partial_\vartheta^2 - \cot \vartheta \partial_\vartheta + \frac{m'^2 + n^2 + \frac{1}{4} - 2m'n \cos \vartheta}{\sin^2 \vartheta} + \frac{m' \cos \vartheta - n}{\sin^2 \vartheta} \sigma_1 \right) P = \\ \left(\lambda^2 + n^2 - \frac{1}{4} \right) P, \quad (6.13)$$

wherefrom all the above results concerning spectrum of \mathcal{D}_n can be derived [26].

6.2 U(1)-case

We should insert the confining solutions (2.6) into Dirac equation (5.4), where Ψ will be just a four-dimensional Dirac spinor, and let us employ the ansatz

$$\Psi = e^{i\omega t} r^{-1} \begin{pmatrix} F_1(r) \Phi(\vartheta, \varphi) \\ F_2(r) \sigma_1 \Phi(\vartheta, \varphi) \end{pmatrix}, \quad (6.14)$$

with a 2D spinor $\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$. Then, after a simple matrix algebra computation, we can get from (5.4) the system

$$\begin{aligned} \left[\left(\partial_r + \frac{1}{r} \right) + \frac{1}{r} \mathcal{D}_0 - \frac{\sigma_2}{\sin \vartheta} g \left(b + \frac{B}{r} \right) \right] \frac{1}{r} F_1 \Phi &= i(\mu_0 - c) \frac{1}{r} F_2 \Phi, \\ \left[\left(\partial_r + \frac{1}{r} \right) + \frac{1}{r} \mathcal{D}_0 - \frac{\sigma_2}{\sin \vartheta} g \left(b + \frac{B}{r} \right) \right] \frac{1}{r} F_2 \sigma_1 \Phi &= -i(\mu_0 + c) \frac{1}{r} F_1 \sigma_1 \Phi \end{aligned} \quad (6.15)$$

with $c = \omega - g(-a/r + A)$ while the euclidean Dirac operator \mathcal{D}_0 on the unit sphere \mathbb{S}^2 is given by (6.5) at $n = 0$. It is not complicated to check that at $b \neq 0$, $B \neq 0$ the variables r and ϑ are not separated. Under this situation we shall at first restrict ourselves to the case $b = B = 0$ since under the circumstances we can solve Eq. (5.4) exactly. Really we employ the ansatz (6.14) and obtain the system (due to the fact that $\sigma_1 \mathcal{D}_0 = -\mathcal{D}_0 \sigma_1$)

$$\begin{aligned} \left(\partial_r + \frac{\lambda}{r} \right) F_1 &= i(\mu_0 - c) F_2, \\ \left(\partial_r - \frac{\lambda}{r} \right) F_2 &= -i(\mu_0 + c) F_1 \end{aligned} \quad (6.16)$$

with an eigenvalue λ for the eigenspinor Φ of the above operator \mathcal{D}_0 , $\lambda = \pm(l + 1) \in \mathbb{Z} \setminus \{0\}$, $l = 0, 1, 2, \dots$ (see previous Subsection).

Let us now employ the ansatz

$$\begin{aligned} F_1 &= \sqrt{\mu_0 - (\omega - gA)} r^\alpha e^{-\beta r} [f_1(x) + f_2(x)], \\ F_2 &= i \sqrt{\mu_0 + (\omega - gA)} r^\alpha e^{-\beta r} [f_1(x) - f_2(x)] \end{aligned} \quad (6.17)$$

with $\alpha = \sqrt{\lambda^2 - g^2 a^2}$, $\beta = \sqrt{\mu_0^2 - (\omega - gA)^2}$, $x = 2\beta r$.

Then, inserting the ansatz into (6.16), adding and subtracting equations give rise to

$$\beta x f_1' + Y f_1 + Z f_2 = 0, \quad (6.18a)$$

$$\beta x f_2' - \beta x f_2 + Y_0 f_2 + Z_0 f_1 = 0, \quad (6.18b)$$

where prime signifies the differentiation with respect to x , $Y, Y_0 = \alpha\beta \mp ga(\omega - gA)$, $Z, Z_0 = \lambda\beta \pm ga\mu_0$. From (6.18), if using the relations $Y Y_0 - Z Z_0 = 0$, $Y + Y_0 = 2\alpha\beta$, one yields the second order equations in x

$$x f_1'' + (1 + 2\alpha - x) f_1' - \frac{Y}{\beta} f_1 = 0, \quad (6.19a)$$

$$x f_2'' + (1 + 2\alpha - x) f_2' - \left(1 + \frac{Y}{\beta} \right) f_2 = 0, \quad (6.19b)$$

that are the Kummer equations (confluent hypergeometric equations in another terminology, see, e. g. Ref.[29]) and for (6.19a) the only finite solution at 0 and at infinity not strongly

increasing is the Laguerre polynomial $L_n^\rho(x)$ with $n = -Y/\beta = 0, 1, 2, \dots$. This gives the spectrum

$$\omega = gA \pm \mu_0 \left[1 + \frac{g^2 a^2}{(n + \sqrt{\lambda^2 - g^2 a^2})^2} \right]^{-1/2}, \quad (6.20)$$

wherefrom it is clear that constant A only shift the origin of count for energy and we can consider $A = 0$. Further, putting $f_1 = CL_n^{2\alpha}(x)$ with some constant C , from (6.18a) at $n > 0$ we find

$$f_2 = \frac{C}{Z} \left[\beta x L_{n-1}^{2\alpha+1}(x) - Y L_n^{2\alpha}(x) \right]$$

because $[L_n^{2\alpha}(x)]' = -L_{n-1}^{2\alpha+1}(x)$ [28], that entails

$$\begin{aligned} F_1 &= C\sqrt{\mu_0 - \omega} r^\alpha e^{-\beta r} \left[\left(1 - \frac{Y}{Z}\right) L_n^{2\alpha}(x) + \frac{\beta}{Z} x L_{n-1}^{2\alpha+1}(x) \right], \\ F_2 &= iC\sqrt{\mu_0 + \omega} r^\alpha e^{-\beta r} \left[\left(1 + \frac{Y}{Z}\right) L_n^{2\alpha}(x) - \frac{\beta}{Z} x L_{n-1}^{2\alpha+1}(x) \right]. \end{aligned} \quad (6.21)$$

The case $n = 0$ should be considered separately. We here have $Y = 0$, $Y_0 = 2\alpha\beta = 2ga\omega$, $\omega = \pm\mu_0\sqrt{\lambda^2 - g^2 a^2}/|\lambda|$, $f_1 = CL_0^{2\alpha}(x) = C$. Further $ZZ_0 = 0 = (\lambda\beta)^2 - (ga\mu_0)^2$ which entails $|\lambda|\beta = g|a|\mu_0$. Then at $a > 0, \lambda > 0$ we get $Z = \lambda\beta + ga\mu_0 = |\lambda|\beta + g|a|\mu_0 = 2ga\mu_0 > 0$, $Z_0 = \lambda\beta - ga\mu_0 = |\lambda|\beta - g|a|\mu_0 = 0$, $f_2 = -CY/Z = -CZ_0/Y_0 = 0$ and

$$F_1 = C\sqrt{\mu_0 - \omega} r^\alpha e^{-\beta r}, F_2 = iC\sqrt{\mu_0 + \omega} r^\alpha e^{-\beta r}. \quad (6.22)$$

At $a > 0, \lambda < 0$ we obtain $Z = \lambda\beta + ga\mu_0 = -|\lambda|\beta + g|a|\mu_0 = 0 > 0$, $Z_0 = -2ga\mu_0 < 0$, $f_2 = -CZ_0/Y_0 = C\mu_0/\omega = C|\lambda|/(\pm\sqrt{\lambda^2 - g^2 a^2})$ and

$$F_1 = C\sqrt{\mu_0 - \omega} r^\alpha e^{-\beta r} \left(1 + \frac{\mu_0}{\omega}\right), F_2 = iC\sqrt{\mu_0 + \omega} r^\alpha e^{-\beta r} \left(1 - \frac{\mu_0}{\omega}\right). \quad (6.23)$$

At $a < 0, \lambda < 0$ we get $Z = \lambda\beta + ga\mu_0 = -|\lambda|\beta - g|a|\mu_0 = -2g|a|\mu_0 < 0$, $Z_0 = \lambda\beta - ga\mu_0 = -|\lambda|\beta + g|a|\mu_0 = 0$, $f_2 = 0$ and $F_{1,2}$ are given by (6.22). At last, at $a < 0, \lambda > 0$ we have $Z = \lambda\beta + ga\mu_0 = |\lambda|\beta - g|a|\mu_0 = 0$, $Z_0 = \lambda\beta - ga\mu_0 = |\lambda|\beta + g|a|\mu_0 = 2g|a|\mu_0$, $f_2 = -CZ_0/Y_0 = C\mu_0/\omega$ and $F_{1,2}$ are given by (6.23).

To describe relativistic bound states we should require $\Psi \in L_2^4(\mathbb{R}^3)$ at any $t \in \mathbb{R}$ and one can accept the normalization condition for F_1, F_2 in the form

$$\int_0^\infty (|F_1|^2 + |F_2|^2) dr = 1 \quad (6.24)$$

with taking into account the condition (6.7) so that C can be determined from the relation (6.24).

It is the expression (6.20) that is in essence the Sommerfeld formula mentioned in Subsection 5.1. It should be noted, however, that standard parametrization in the Sommerfeld

formula adduced in all the monographs (see, e. g., [23, 25, 30]) is somewhat different from that in formula (6.20). The fact is that the standard approach uses the orthonormal basis in $L^2_2(\mathbb{S}^2)$ different from the basis of the eigenspinors of the Dirac operator \mathcal{D}_0 . Namely, one uses the eigenbasis of the operator $K = \vec{\sigma}\vec{L} + 1$, where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and $\vec{L} = -i(\vec{r} \times \partial/\partial\vec{r})$ is the angular momentum operator. The eigenvalues of operator K are numbers of the form $\lambda' = \pm(j + 1/2)$ with $j = 1/2, 3/2, \dots$, so j defines the total angular momentum $\vec{J} = \vec{L} + \vec{S}$ with spin momentum $\vec{S} = \vec{\sigma}/2$, that is J^2 has the eigenvalues $j(j + 1)$. In standard approach just λ' stands in (6.20) instead of λ . It is evident that both formulas (with λ or λ') reproduce the same spectrum but the corresponding wave functions will be slightly different in angular part depending on ϑ, φ since the Dirac operator \mathcal{D}_0 does not commute with operator K . There are at least two reasons why historically operator K was employed rather than \mathcal{D}_0 . The first one is that in standard approach one solves Dirac equation (5.4) in Cartesian coordinates without transition to spherical ones and under this situation there naturally arises just operator K whereas when passing on to the spherical coordinates there naturally would arise just operator \mathcal{D}_0 while separating variables. The second reason is that the whole formula (6.20) is not necessary in nonrelativistic systems such as hydrogen atom, positronium and so on. It is enough to restrict themselves to a few terms of expansion in g^2 for (6.20) to obtain corrections (fine structure) to the nonrelativistic spectrum which are in concordance with experiment and one may use the notion of spin as essentially nonrelativistic phenomenon. Another matter are quarks in mesons that should probably be considered essentially relativistic objects and it is the most natural to write down Dirac equation (5.4) in Minkowski spacetime with spherical coordinates in its spatial part. It is the latter way that we pursue in present paper.

At the end of this Subsection we can slightly generalize the results obtained if inserting solution (5.19) at $b = B = 0$ into Dirac equation (5.4). Then it is not complicated to see we shall get the similar relations providing that $K = k/g$ with $k \in \mathbb{Z}$, i. e., k is integer number. But now we should consider the spinor Φ of (6.14) to be the eigenspinor of the twisted euclidean Dirac operator \mathcal{D}_k on the unit sphere \mathbb{S}^2 with the Chern number k (see Subsection 6.1) and the eigenvalues λ should be, accordingly, replaced by $\lambda = \pm\sqrt{(l+1)^2 - k^2}$, $l \geq |k|$. Physically the corresponding configurations describe the Dirac monopole ones with magnetic charge $P = k/g$ so the corresponding wave functions should be modified in obvious way.

6.3 $N = 2$

After inserting the solution (5.21) (with $A = 0$) into (5.8) for both the equations we employ the ansatz

$$\Psi_j = e^{i\omega_j t} r^{-1} \begin{pmatrix} F_{j1}(r)\Phi_j(\vartheta, \varphi) \\ F_{j2}(r)\sigma_1\Phi_j(\vartheta, \varphi) \end{pmatrix}, j = 1, 2 \quad (6.25)$$

with a 2D spinor $\Phi_j = \begin{pmatrix} \Phi_{j1} \\ \Phi_{j2} \end{pmatrix}$ which entails the systems

$$\left(\partial_r + \frac{\lambda_j}{r}\right) F_{j1} = i(\mu_0 - c_j) F_{j2},$$

$$\left(\partial_r - \frac{\lambda_j}{r}\right) F_{j2} = -i(\mu_0 + c_j) F_{j1} \quad (6.26)$$

with an eigenvalue λ_j for the eigenspinor Φ_j of the Dirac operator \mathcal{D}_0 , $\lambda_j = \pm(l_j + 1) \in \mathbb{Z} \setminus \{0\}$, $l_j = 0, 1, 2, \dots$. Besides

$$c_1 = \omega_1 + ga/r, c_2 = \omega_2 - ga/r, \quad (6.27)$$

so that the energy spectrum ω of particle is given by the relation $\omega = \omega_1 + \omega_2$. Acting along the same lines as in previous Subsection we obtain the spectrum of particle in the form

$$\frac{\omega}{\mu_0} = \pm \left[1 + \frac{g^2 a^2}{(n_1 + \sqrt{\lambda_1^2 - g^2 a^2})^2} \right]^{-1/2} \pm \left[1 + \frac{g^2 a^2}{(n_2 + \sqrt{\lambda_2^2 - g^2 a^2})^2} \right]^{-1/2}, \quad (6.28)$$

where the number $n_{1,2} = 0, 1, 2, \dots$

If $K \neq 0$ in (5.25) at $b = B = 0$ then when inserting (5.21), (5.25) into (5.8) we shall get the similar spectrum (6.28) providing that $K = k/g$ with $k \in \mathbb{Z}$, i. e., k is an integer number. But now we should consider the spinor Φ_j of (6.25) to be the eigenspinor Φ_j of the twisted euclidean Dirac operators $\mathcal{D}_{\pm k}$ on the unit sphere \mathbb{S}^2 (see Subsection 6.1), respectively, with the Chern numbers $\pm k$ and the eigenvalues λ_j should be, accordingly, replaced by $\lambda_1 = \pm\sqrt{(l_1 + 1)^2 - k^2}$, $l_1 \geq |k|$, $\lambda_2 = \pm\sqrt{(l_2 + 1)^2 - k^2}$, $l_2 \geq |k|$. Physically the corresponding configurations of SU(2)-field describe the Dirac-like monopole ones with magnetic charges, conformably, $P_1 = k/g$, $P_2 = -k/g$ but the total (nonabelian) magnetic charge of the given configurations remains equal to $P_1 + P_2 = 0$.

The corresponding wave functions are not complicated to be written out on the analogy of U(1)-case of previous Subsection but we shall not dwell upon it. It should be only noted that the conditon (6.24) should be replaced by

$$\int_0^\infty (|F_{j1}|^2 + |F_{j2}|^2) dr = \frac{1}{2}, \quad j = 1, 2.$$

6.4 $N = 3$

We here use the solution for A_t of (2.13) (with $A_1 = A_2 = 0$) and for all three equations in (5.9) we employ the ansatz

$$\Psi_j = e^{i\omega_j t} r^{-1} \begin{pmatrix} F_{j1}(r) \Phi_j(\vartheta, \varphi) \\ F_{j2}(r) \sigma_1 \Phi_j(\vartheta, \varphi) \end{pmatrix}, \quad j = 1, 2, 3 \quad (6.29)$$

with a 2D spinor $\Phi_j = \begin{pmatrix} \Phi_{j1} \\ \Phi_{j2} \end{pmatrix}$ which entails the systems

$$\begin{aligned} \left(\partial_r + \frac{\lambda_j}{r}\right) F_{j1} &= i(\mu_0 - c_j) F_{j2}, \\ \left(\partial_r - \frac{\lambda_j}{r}\right) F_{j2} &= -i(\mu_0 + c_j) F_{j1} \end{aligned} \quad (6.30)$$

with an eigenvalue λ_j for the eigenspinor Φ_j of the Dirac operator \mathcal{D}_0 , $\lambda_j = \pm(l_j + 1) \in \mathbb{Z} \setminus \{0\}$, $l_j = 0, 1, 2, \dots$. Besides

$$c_1 = \omega_1 + ga_1/r, c_2 = \omega_2 + ga_2/r, c_3 = \omega_3 - g(a_1 + a_2)/r, \quad (6.31)$$

so that the energy spectrum ω of particle is given by the relation $\omega = \omega_1 + \omega_2 + \omega_3$ which we obtain in the form

$$\begin{aligned} \frac{\omega}{\mu_0} = \pm & \left[1 + \frac{g^2 a_1^2}{(n_1 + \sqrt{\lambda_1^2 - g^2 a_1^2})^2} \right]^{-1/2} \pm \left[1 + \frac{g^2 a_2^2}{(n_2 + \sqrt{\lambda_2^2 - g^2 a_2^2})^2} \right]^{-1/2} \\ & \pm \left[1 + \frac{g^2 (a_1 + a_2)^2}{(n_3 + \sqrt{\lambda_3^2 - g^2 (a_1 + a_2)^2})^2} \right]^{-1/2}, \end{aligned} \quad (6.32)$$

where the number $n_{1,2,3} = 0, 1, 2, \dots$

If $K_j \neq 0$ in (5.26) at $b_j = B_j = 0$ then when inserting (2.13), (5.26) into (5.9) we shall get the similar spectrum (6.32) providing that $K_j = k_j/g$ with $k_j \in \mathbb{Z}$, i. e., k_j are integers. But now we should consider the spinor Φ_j of (6.29) to be the eigenspinor Φ_j of the twisted euclidean Dirac operator \mathcal{D}_k on the unit sphere \mathbb{S}^2 (see Subsection 6.1), respectively, with the Chern numbers $k = k_1, k_2, -(k_1 + k_2)$ and the eigenvalues λ_j should be, accordingly, replaced by $\lambda_1 = \pm\sqrt{(l_1 + 1)^2 - k_1^2}$, $l_1 \geq |k_1|$, $\lambda_2 = \pm\sqrt{(l_2 + 1)^2 - k_2^2}$, $l_2 \geq |k_2|$, $\lambda_3 = \pm\sqrt{(l_3 + 1)^2 - (k_1 + k_2)^2}$, $l_3 \geq |k_1 + k_2|$. The corresponding configurations of gluonic field describe the Dirac-like monopole ones with magnetic charges, conformably, $P_1 = k_1/g$, $P_2 = k_2/g$, $P_3 = -(k_1 + k_2)/g$, but the total (nonabelian) magnetic charge of the given configurations remains equal to $P_1 + P_2 + P_3 = 0$.

The corresponding wave functions are again not complicated to be written out on the analogy of U(1)-case of Subsection 6.2 but we do not dwell upon it and the condition (6.24) should be replaced by

$$\int_0^\infty (|F_{j1}|^2 + |F_{j2}|^2) dr = \frac{1}{3}, j = 1, 2, 3.$$

6.5 $N = 4$

If inserting the solution for A_t of (5.27) (with $A_1 = A_2 = A_3 = 0$) into (5.10) then for all four equations in (5.10) the ansatz of form (6.29) with $j = 1, 2, 3, 4$ will lead to the systems of form (6.30) with an eigenvalue λ_j for the eigenspinor Φ_j of the Dirac operator \mathcal{D}_0 , where $\lambda_j = \pm(l_j + 1) \in \mathbb{Z} \setminus \{0\}$, $l_j = 0, 1, 2, \dots$, while

$$c_1 = \omega_1 + ga_1/r, c_2 = \omega_2 + ga_2/r, c_3 = \omega_3 + ga_3/r, c_4 = \omega_4 - g(a_1 + a_2 + a_3)/r, \quad (6.33)$$

so that the energy spectrum ω of particle is given by the relation $\omega = \omega_1 + \omega_2 + \omega_3 + \omega_4$ and is obtained in the form

$$\frac{\omega}{\mu_0} = \pm \left[1 + \frac{g^2 a_1^2}{(n_1 + \sqrt{\lambda_1^2 - g^2 a_1^2})^2} \right]^{-1/2} \pm \left[1 + \frac{g^2 a_2^2}{(n_2 + \sqrt{\lambda_2^2 - g^2 a_2^2})^2} \right]^{-1/2}$$

$$\pm \left[1 + \frac{g^2 a_3^2}{(n_3 + \sqrt{\lambda_3^2 - g^2 a_3^2})^2} \right]^{-1/2} \pm \left[1 + \frac{g^2 (a_1 + a_2 + a_3)^2}{(n_4 + \sqrt{\lambda_4^2 - g^2 (a_1 + a_2 + a_3)^2})^2} \right]^{-1/2}, \quad (6.34)$$

where the number $n_{1,2,3,4} = 0, 1, 2, \dots$

If $K_j \neq 0$ in (5.28) at $b_j = B_j = 0$ then when inserting (5.27), (5.28) into (5.10) we shall get the similar spectrum (6.34) providing that $K_j = k_j/g$ with $k_j \in \mathbb{Z}$, i. e., k_j are integers, so the spinor Φ_j is the eigenspinor Φ_j of the twisted euclidean Dirac operator \mathcal{D}_k on the unit sphere \mathbb{S}^2 (see Subsection 6.1), respectively, with the Chern numbers $k = k_1, k_2, k_3, -(k_1 + k_2 + k_3)$ and the eigenvalues λ_j should be, accordingly, replaced by $\lambda_1 = \pm \sqrt{(l_1 + 1)^2 - k_1^2}$, $l_1 \geq |k_1|$, $\lambda_2 = \pm \sqrt{(l_2 + 1)^2 - k_2^2}$, $l_2 \geq |k_2|$, $\lambda_3 = \pm \sqrt{(l_3 + 1)^2 - k_3^2}$, $l_3 \geq |k_3|$, $\lambda_4 = \pm \sqrt{(l_4 + 1)^2 - (k_1 + k_2 + k_3)^2}$, $l_4 \geq |k_1 + k_2 + k_3|$. The corresponding configurations of SU(4)-field describe the Dirac-like monopole ones with magnetic charges, conformably, $P_1 = k_1/g$, $P_2 = k_2/g$, $P_3 = k_3/g$, $P_4 = -(k_1 + k_2 + k_3)/g$ but as before the total (nonabelian) magnetic charge of the given configurations remains equal to $P_1 + P_2 + P_3 + P_4 = 0$.

The corresponding wave functions are again not complicated to be written out on the analogy of U(1)-case of Subsection 6.2 but we do not dwell upon it while the condition (6.24) should be replaced by

$$\int_0^\infty (|F_{j1}|^2 + |F_{j2}|^2) dr = \frac{1}{4}, j = 1, 2, 3, 4.$$

7 Spectrum of Bound States in the Coulomb-Linear Case

7.1 U(1)-case

We now should return to the system (6.15) at $b \neq 0, B \neq 0, A = 0$. One may hope to obtain the almost exact solution of (6.15) if considering $\sigma_2 \Phi \approx \sin \vartheta \Phi$. As follows from the estimate in Appendix D this condition is rather good fulfilled and when doing so we probably make an error retaining eigenvalues λ of the euclidean Dirac operator \mathcal{D}_0 on the unit sphere \mathbb{S}^2 instead of the eigenvalues of a less symmetric operator on \mathbb{S}^2 whose form is unknown explicitly.

Having accepted the mentioned condition we come to the system (owing to the fact that $\sigma_1 \mathcal{D}_0 = -\mathcal{D}_0 \sigma_1$, $\sigma_2 \sigma_1 = -\sigma_1 \sigma_2$)

$$\begin{aligned} \left[\partial_r + \frac{\lambda}{r} - g \left(b + \frac{B}{r} \right) \right] F_1 &= i(\mu_0 - c) F_2, \\ \left[\partial_r - \frac{\lambda}{r} + g \left(b + \frac{B}{r} \right) \right] F_2 &= -i(\mu_0 + c) F_1 \end{aligned} \quad (7.1)$$

with $c = \omega + ga/r$.

Now we employ the ansatz

$$F_1 = Pr^\alpha e^{-\beta r} [f_1(x) + f_2(x)], F_2 = iQr^\alpha e^{-\beta r} [f_1(x) - f_2(x)] \quad (7.1')$$

with $\alpha = \sqrt{(\lambda - gB)^2 - g^2 a^2}$, $\beta = \sqrt{\mu_0^2 - \omega^2 + g^2 b^2}$, $P = gb + \beta$, $Q = \mu_0 + \omega$, $x = 2\beta r$.

After this, inserting the ansatz into (7.1), adding and subtracting equations entail

$$xPQf_1' + Yf_1 + Zf_2 = 0, \quad (7.2a)$$

$$xPQf_2' - xPQf_2 + Y_0f_2 + \left(Z_0 - \frac{gb}{\beta}PQx\right)f_1 = 0, \quad (7.2b)$$

where prime signifies the differentiation with respect to x , $Y, Y_0 = [\alpha\beta \mp gaw + g\alpha b]Q \pm g^2 abP$, $Z, Z_0 = [(\lambda - gB)P \pm ga\mu_0]Q \pm g^2 abP$ and $YY_0 - ZZ_0 = 0$.

From (7.2a)–(7.2b) one yields the second order equations in x

$$xf_1'' + (1 + 2\alpha - x)f_1' + nf_1 = 0, \quad (7.3)$$

$$xf_2'' + \left(\frac{Z_0}{Z_0 - b_0x} + 2\alpha - x\right)f_2' + n\left(\frac{Z_0\kappa}{Z_0 - b_0x} + 1\right)f_2 = 0 \quad (7.4)$$

with $b_0 = gbPQ/\beta$, $\kappa = PQ/Y$ and

$$n = \frac{gbZ - \beta Y}{\beta PQ}, \quad (7.5)$$

which entails the equation for spectrum of ω

$$\begin{aligned} &[g^2 a^2 + (n + \alpha)^2]\omega^2 + 2(\lambda - gB)g^2 ab\omega + \\ &[(\lambda - gB)^2 - (n + \alpha)^2]g^2 b^2 - \mu_0^2(n + \alpha)^2 = 0, \end{aligned} \quad (7.6)$$

that yields

$$\omega = \frac{-(\lambda - gB)g^2 ab \pm \sqrt{X}}{g^2 a^2 + (n + \alpha)^2} \quad (7.7)$$

with $X = (\lambda - gB)^2 g^4 a^2 b^2 - [g^2 a^2 + (n + \alpha)^2]\{[(\lambda - gB)^2 - (n + \alpha)^2]g^2 b^2 - \mu_0^2(n + \alpha)^2\}$ and it is clear that the expression (7.7) passes on to (6.20) at $b = B = 0$ while (7.7) can be rewritten in a more symmetrical form

$$\omega = \omega(n, l, \lambda) = \frac{-\Lambda g^2 ab \pm (n + \alpha)\sqrt{(n^2 + 2n\alpha + \Lambda^2)\mu_0^2 + g^2 b^2(n^2 + 2n\alpha)}}{n^2 + 2n\alpha + \Lambda^2} \quad (7.8)$$

with $\Lambda = \lambda - gB = \pm(l + 1) - gB$.

It is clear that according to (7.3) (which is the confluent hypergeometric equation) we should choose $f_1 = CL_n^{2\alpha}(x)$ with the Laguerre polynomial $L_n^{2\alpha}(x)$ and some constant C if $n = 0, 1, 2, \dots$. Then at $n > 0$ from (7.2a) we find

$$f_2 = \frac{C}{Z} [xPQL_{n-1}^{2\alpha+1}(x) - YL_n^{2\alpha}(x)]$$

because $[L_n^{2\alpha}(x)]' = -L_{n-1}^{2\alpha+1}(x)$ [28], that entails

$$\begin{aligned} F_1 &= CP r^\alpha e^{-\beta r} \left[\left(1 - \frac{Y}{Z}\right) L_n^{2\alpha}(x) + \frac{PQ}{Z} x L_{n-1}^{2\alpha+1}(x) \right], \\ F_2 &= iCQ r^\alpha e^{-\beta r} \left[\left(1 + \frac{Y}{Z}\right) L_n^{2\alpha}(x) - \frac{PQ}{Z} x L_{n-1}^{2\alpha+1}(x) \right]. \end{aligned} \quad (7.9)$$

At $n = 0$ we have $f_1 = CL_0^{2\alpha}(x) = C$, $f_2 = -CY/Z$, wherefrom

$$\begin{aligned} F_1 &= CP r^\alpha e^{-\beta r} \left(1 - \frac{Y}{Z}\right) = CP r^\alpha e^{-\beta r} \left(1 - \frac{gb}{\beta}\right), \\ F_2 &= iCQ r^\alpha e^{-\beta r} \left(1 + \frac{Y}{Z}\right) = iCQ r^\alpha e^{-\beta r} \left(1 + \frac{gb}{\beta}\right), \end{aligned} \quad (7.10)$$

inasmuch as $gbZ = \beta Y$ at $n = 0$ according to (7.5).

Hence we can see that Ψ of (6.14) $\in L_2^4(\mathbb{R}^3)$ at any $t \in \mathbb{R}$ and, conformably, Ψ may describe relativistic bound states with the energy spectrum (7.8).

Also it should be noted that the influence of the Dirac monopole configurations for U(1)-field when $K \neq 0$ in (5.19) can be treated by the same manner as in Subsection 6.2 if taking $\sigma_2 \Phi \approx \sin \vartheta \Phi$ for the eigenspinor Φ of the twisted euclidean Dirac operator \mathcal{D}_k on the unit sphere \mathbb{S}^2 with the conforming Chern number k .

At last, constant C from (7.9)–(7.10) is defined by the condition (6.24) as before.

7.2 $N = 2$

The corresponding modifications of U(1)-case here are obvious so we shall only briefly describe them. The spectrum is given by $\omega = \omega_1 + \omega_2$ with

$$\begin{aligned} \omega_1 &= \omega_1(n_1, l_1, \lambda_1) = \\ &= \frac{-\Lambda_1 g^2 ab \pm (n_1 + \alpha_1) \sqrt{(n_1^2 + 2n_1 \alpha_1 + \Lambda_1^2) \mu_0^2 + g^2 b^2 (n_1^2 + 2n_1 \alpha_1)}}{n_1^2 + 2n_1 \alpha_1 + \Lambda_1^2}, \end{aligned} \quad (7.11)$$

$$\begin{aligned} \omega_2 &= \omega_2(n_2, l_2, \lambda_2) = \\ &= \frac{-\Lambda_2 g^2 ab \pm (n_2 + \alpha_2) \sqrt{(n_2^2 + 2n_2 \alpha_2 + \Lambda_2^2) \mu_0^2 + g^2 b^2 (n_2^2 + 2n_2 \alpha_2)}}{n_2^2 + 2n_2 \alpha_2 + \Lambda_2^2} \end{aligned} \quad (7.12)$$

with $\Lambda_1 = \lambda_1 - gB = \pm(l_1 + 1) - gB$, $\Lambda_2 = \lambda_2 + gB = \pm(l_2 + 1) + gB$, $\alpha_1 = \sqrt{\Lambda_1^2 - g^2 a^2}$, $\alpha_2 = \sqrt{\Lambda_2^2 - g^2 a^2}$. The corresponding radial parts of wave functions (6.25) are given at $n_j = 0$ by

$$F_{j1} = C_j P_j r^{\alpha_j} e^{-\beta_j r} \left(1 - \frac{Y_j}{Z_j}\right), F_{j2} = iC_j Q_j r^{\alpha_j} e^{-\beta_j r} \left(1 + \frac{Y_j}{Z_j}\right), \quad (7.13)$$

while at $n_j > 0$ by

$$\begin{aligned} F_{j1} &= C_j P_j r^{\alpha_j} e^{-\beta_j r} \left[\left(1 - \frac{Y_j}{Z_j} \right) L_{n_j}^{2\alpha_j}(r_j) + \frac{P_j Q_j}{Z_j} r_j L_{n_j-1}^{2\alpha_j+1}(r_j) \right], \\ F_{j2} &= i C_j Q_j r^{\alpha_j} e^{-\beta_j r} \left[\left(1 + \frac{Y_j}{Z_j} \right) L_{n_j}^{2\alpha_j}(r_j) - \frac{P_j Q_j}{Z_j} r_j L_{n_j-1}^{2\alpha_j+1}(r_j) \right] \end{aligned} \quad (7.14)$$

with the Laguerre polynomials $L_{n_j}^\rho(r_j)$, $r_j = 2\beta_j r$, where $\beta_j = \sqrt{\mu_0^2 - \omega_j^2 + g^2 b^2}$, $P_1 = gb + \beta_1$, $P_2 = -gb + \beta_2$, $Q_j = \mu_0 + \omega_j$, $Y_1 = (\alpha_1 \beta_1 - ga\omega_1 + g\alpha_1 b)Q_1 + g^2 ab P_1$, $Y_2 = (\alpha_2 \beta_2 + ga\omega_2 - g\alpha_2 b)Q_2 + g^2 ab P_2$, $Z_1 = [(\lambda_1 - gB)P_1 + ga\mu_0]Q_1 + g^2 ab P_1$, $Z_2 = [(\lambda_2 + gB)P_2 - ga\mu_0]Q_2 + g^2 ab P_2$. Also it should be noted that the quantum numbers n_j are defined by the relations

$$n_1 = \frac{gbZ_1 - \beta_1 Y_1}{\beta_1 P_1 Q_1}, n_2 = -\frac{gbZ_2 + \beta_2 Y_2}{\beta_2 P_2 Q_2}, \quad (7.15)$$

Further, C_j is determined from the normalization condition

$$\int_0^\infty (|F_{j1}|^2 + |F_{j2}|^2) dr = \frac{1}{2}. \quad (7.16)$$

Consequently, we shall gain that in (6.25) $\Psi_j \in L_2^4(\mathbb{R}^3)$ at any $t \in \mathbb{R}$ and, as a result, $\Psi = (\Psi_1, \Psi_2)$ may describe relativistic bound states with the energy spectrum (7.11)–(7.12).

Finally, it should be noted that the influence of the Dirac-like monopole configurations for SU(2)-field when $K \neq 0$ in (5.25) can be treated by the same manner as in Subsection 6.3 if taking $\sigma_2 \Phi_j \approx \sin \vartheta \Phi_j$ for the eigenspinor Φ_j of the twisted euclidean Dirac operator $\mathcal{D}_{\pm k}$ on the unit sphere \mathbb{S}^2 with the conforming Chern numbers $\pm k$.

7.3 $N = 3$

Let us adduce the coresponding results without going into details that can be easily reconstructed along the lines of Subsection 7.1. The spectrum is given by $\omega = \omega_1 + \omega_2 + \omega_3$ with

$$\begin{aligned} \omega_1 &= \omega_1(n_1, l_1, \lambda_1) = \\ &= \frac{-\Lambda_1 g^2 a_1 b_1 \pm (n_1 + \alpha_1) \sqrt{(n_1^2 + 2n_1 \alpha_1 + \Lambda_1^2) \mu_0^2 + g^2 b_1^2 (n_1^2 + 2n_1 \alpha_1)}}{n_1^2 + 2n_1 \alpha_1 + \Lambda_1^2}, \end{aligned} \quad (7.17)$$

$$\begin{aligned} \omega_2 &= \omega_2(n_2, l_2, \lambda_2) = \\ &= \frac{-\Lambda_2 g^2 a_2 b_2 \pm (n_2 + \alpha_2) \sqrt{(n_2^2 + 2n_2 \alpha_2 + \Lambda_2^2) \mu_0^2 + g^2 b_2^2 (n_2^2 + 2n_2 \alpha_2)}}{n_2^2 + 2n_2 \alpha_2 + \Lambda_2^2}, \end{aligned} \quad (7.18)$$

$$\omega_3 = \omega_3(n_3, l_3, \lambda_3) =$$

$$\frac{-\Lambda_3 g^2 a_3 b_3 \pm (n_3 + \alpha_3) \sqrt{(n_3^2 + 2n_3 \alpha_3 + \Lambda_3^2) \mu_0^2 + g^2 b_3^2 (n_3^2 + 2n_3 \alpha_3)}}{n_3^2 + 2n_3 \alpha_3 + \Lambda_3^2}, \quad (7.19)$$

where $a_3 = -(a_1 + a_2)$, $b_3 = -(b_1 + b_2)$, $\Lambda_j = \lambda_j - gB_j$, $j = 1, 2, 3$, $B_3 = -(B_1 + B_2)$, $n_j = 0, 1, 2, \dots$, while $\lambda_j = \pm(l_j + 1)$ are the eigenvalues of euclidean Dirac operator \mathcal{D}_0 on unit sphere with $l_j = 0, 1, 2, \dots$. Besides

$$\alpha_1 = \sqrt{\Lambda_1^2 - g^2 a_1^2}, \alpha_2 = \sqrt{\Lambda_2^2 - g^2 a_2^2}, \alpha_3 = \sqrt{\Lambda_3^2 - g^2 (a_1 + a_2)^2}. \quad (7.20)$$

Further, radial parts of (6.29) are given at $n_j = 0$ by (7.13) while at $n_j > 0$ by (7.14). We have $r_j = 2\beta_j r$, where $\beta_j = \sqrt{\mu_0^2 - \omega_j^2 + g^2 b_j^2}$ at $j = 1, 2, 3$ with $b_3 = -(b_1 + b_2)$, $P_j = gb_j + \beta_j$, $j = 1, 2$, $P_3 = -g(b_1 + b_2) + \beta_3$, $Q_j = \mu_0 + \omega_j$, $Y_j = (\alpha_j \beta_j - ga_j \omega_j + g\alpha_j b_j)Q_j + g^2 a_j b_j P_j$, $j = 1, 2$, $Y_3 = [\alpha_3 \beta_3 + g(a_1 + a_2)\omega_3 - g\alpha_3(b_1 + b_2)]Q_3 + g^2(a_1 + a_2)(b_1 + b_2)P_3$, $Z_j = [(\lambda_j - gB_j)P_j + ga_j \mu_0]Q_j + g^2 a_j b_j P_j$, $j = 1, 2$, $Z_3 = [(\lambda_3 + g(B_1 + B_2))P_3 - g(a_1 + a_2)\mu_0]Q_3 + g^2(a_1 + a_2)(b_1 + b_2)P_3$, quantum numbers n_j are defined by the relations

$$n_1 = \frac{gb_1 Z_1 - \beta_1 Y_1}{\beta_1 P_1 Q_1}, n_2 = \frac{gb_2 Z_2 - \beta_2 Y_2}{\beta_2 P_2 Q_2}, n_3 = -\frac{g(b_1 + b_2)Z_3 + \beta_3 Y_3}{\beta_3 P_3 Q_3}. \quad (7.21)$$

Further, C_j of (7.13)–(7.14) is determined from the normalization condition

$$\int_0^\infty (|F_{j1}|^2 + |F_{j2}|^2) dr = \frac{1}{3}. \quad (7.21')$$

As a consequence, we shall gain that in (6.29) $\Psi_j \in L_2^4(\mathbb{R}^3)$ at any $t \in \mathbb{R}$ and, accordingly, $\Psi = (\Psi_1, \Psi_2, \Psi_3)$ may describe relativistic bound states with the energy spectrum (7.17)–(7.19).

Finally, it should be noted that the influence of the Dirac-like monopole configurations for gluonic SU(3)-field when $K_j \neq 0$ in (5.26) can be treated by the same manner as in Subsection 6.4 if taking $\sigma_2 \Phi_j \approx \sin \vartheta \Phi_j$ for the eigenspinor Φ_j of the twisted euclidean Dirac operator \mathcal{D}_k on the unit sphere \mathbb{S}^2 with the conforming Chern numbers $k = k_1, k_2, -(k_1 + k_2)$.

7.4 $N = 4$

In line with previous Subsection it is not difficult to write out the results for the given case. The spectrum is given by $\omega = \omega_1 + \omega_2 + \omega_3 + \omega_4$ with

$$\omega_j = \omega_j(n_j, l_j, \lambda_j) = \frac{-\Lambda_j g^2 a_j b_j \pm (n_j + \alpha_j) \sqrt{(n_j^2 + 2n_j \alpha_j + \Lambda_j^2) \mu_0^2 + g^2 b_j^2 (n_j^2 + 2n_j \alpha_j)}}{n_j^2 + 2n_j \alpha_j + \Lambda_j^2}, j = 1, 2, 3, \quad (7.22)$$

$$\omega_4 = \omega_4(n_4, l_4, \lambda_4) = \frac{-\Lambda_4 g^2 a_4 b_4 \pm (n_4 + \alpha_4) \sqrt{(n_4^2 + 2n_4 \alpha_4 + \Lambda_4^2) \mu_0^2 + g^2 b_4^2 (n_4^2 + 2n_4 \alpha_4)}}{n_4^2 + 2n_4 \alpha_4 + \Lambda_4^2}, \quad (7.23)$$

where $a_4 = -(a_1 + a_2 + a_3)$, $b_4 = -(b_1 + b_2 + b_3)$, $\Lambda_j = \lambda_j - gB_j$, $j = 1, 2, 3, 4$, $B_4 = -(B_1 + B_2 + B_3)$, $n_j = 0, 1, 2, \dots$, while $\lambda_j = \pm(l_j + 1)$ are the eigenvalues of euclidean Dirac operator \mathcal{D}_0 on unit sphere with $l_j = 0, 1, 2, \dots$. Besides

$$\alpha_j = \sqrt{\Lambda_j^2 - g^2 a_j^2}, j = 1, 2, 3, \alpha_4 = \sqrt{\Lambda_4^2 - g^2 (a_1 + a_2 + a_3)^2}. \quad (7.24)$$

Further, radial parts of (6.29) are given at $n_j = 0$ by (7.13) while at $n_j > 0$ by (7.14). We have $r_j = 2\beta_j r$, where $\beta_j = \sqrt{\mu_0^2 - \omega_j^2 + g^2 b_j^2}$ at $j = 1, 2, 3, 4$ with $b_4 = -(b_1 + b_2 + b_3)$, $P_j = gb_j + \beta_j$, $j = 1, 2, 3$, $P_4 = -g(b_1 + b_2 + b_3) + \beta_4$, $Q_j = \mu_0 + \omega_j$, $Y_j = (\alpha_j \beta_j - ga_j \omega_j + g\alpha_j b_j)Q_j + g^2 a_j b_j P_j$, $j = 1, 2, 3$, $Y_4 = [\alpha_4 \beta_4 + g(a_1 + a_2 + a_3)\omega_4 - g\alpha_4(b_1 + b_2 + b_3)]Q_4 + g^2(a_1 + a_2 + a_3)(b_1 + b_2 + b_3)P_4$, $Z_j = [(\lambda_j - gB_j)P_j + ga_j \mu_0]Q_j + g^2 a_j b_j P_j$, $j = 1, 2, 3$, $Z_4 = [(\lambda_4 + g(B_1 + B_2 + B_3))P_4 - g(a_1 + a_2 + a_3)\mu_0]Q_4 + g^2(a_1 + a_2 + a_3)(b_1 + b_2 + b_3)P_4$, quantum numbers n_j are defined by the relations

$$n_j = \frac{gb_j Z_j - \beta_j Y_j}{\beta_j P_j Q_j}, j = 1, 2, 3, n_4 = -\frac{g(b_1 + b_2 + b_3)Z_4 + \beta_4 Y_4}{\beta_4 P_4 Q_4}. \quad (7.25)$$

Further, C_j of (7.13)–(7.14) is determined from the normalization condition

$$\int_0^\infty (|F_{j1}|^2 + |F_{j2}|^2) dr = \frac{1}{4}. \quad (7.26)$$

Thus, in (6.29) $\Psi_j \in L_2^4(\mathbb{R}^3)$ at any $t \in \mathbb{R}$ and, consequently, $\Psi = (\Psi_1, \Psi_2, \Psi_3, \Psi_4)$ may describe relativistic bound states with the energy spectrum (7.22)–(7.23).

Finally, it should be noted that the influence of the Dirac-like monopole configurations for gluonic SU(4)-field when $K_j \neq 0$ in (5.28) can be treated by the same manner as in Subsection 6.5 if taking $\sigma_2 \Phi_j \approx \sin \vartheta \Phi_j$ for the eigenspinor Φ_j of the twisted euclidean Dirac operator \mathcal{D}_k on the unit sphere \mathbb{S}^2 with the conforming Chern numbers $k = k_1, k_2, k_3, -(k_1 + k_2 + k_3)$.

7.5 Remark about the case $N > 4$

As is not complicated to see, one can consider general case $N > 4$. To obtain the corresponding confining solutions for given N one should take a concrete realization of the conforming SU(N)-Lie algebra whose form depends on N , as we have seen above. Then the explicit form of the solutions under discussion will also depend on the mentioned realization but it is clear that they will be characterized by real constants a_j, b_j, A_j, B_j , $j = 1, \dots, N$ with $a_N = -(a_1 + a_2 + \dots + a_{N-1})$, $b_N = -(b_1 + b_2 + \dots + b_{N-1})$, $B_N = -(B_1 + B_2 + \dots + B_{N-1})$ while one may put $A_j = 0$.

The shape of spectrum for relativistic bound states will obviously have the same form as in (7.22) with $\Lambda_j = \lambda_j - gB_j$, $n_j = 0, 1, 2, \dots$, but $j = 1, \dots, N$. The corresponding modifications for parameters $\alpha_j, \beta_j, P_j, Q_j$ of the wave functions in the form (7.13)–(7.14) are also evident. The normalization condition (7.26) will contain $1/N$ in its right-hand side. So $\Psi = (\Psi_1, \Psi_2, \dots, \Psi_N)$ may describe relativistic bound states with the energy spectrum $\omega = \omega_1 + \omega_2 + \dots + \omega_N$. At last, the influence of the Dirac-like monopole configurations for gluonic $SU(N)$ -field can be treated by the same manner as in previous Sections.

7.6 Nonrelativistic Limit

It makes sense to obtain the nonrelativistic limit (i.e. when $c \rightarrow \infty$) for spectrum (7.8) in order to us to have a possibility of estimating the contribution of relativistic effects. Changing $a \rightarrow a/(\hbar c)$, $B \rightarrow B/(\hbar c)$ and expanding (7.8) in $z = 1/c$, we get

$$\omega(n, l, \lambda) = \pm \mu_0 c^2 \left[1 \mp \frac{g^2 a^2}{2\hbar^2(n + |\lambda|)^2} z^2 \right] - \left[\frac{\lambda g^2 a b}{\hbar(n + |\lambda|)^2} \pm \mu_0 \frac{g^3 B a^2 f(n, \lambda)}{\hbar^3(n + |\lambda|)^{13}} \right] z + O(z^2), \quad (7.27)$$

where

$$f(n, \lambda) = 10n^9\lambda + 120n^7\lambda^3 + 252n^5\lambda^5 + 120n^3\lambda^7 + 10n\lambda^9 + \frac{|\lambda|}{\lambda} \left(n^{10} + 45n^8\lambda^2 + 210n^6\lambda^4 + 210n^4\lambda^6 + 45n^2\lambda^8 + \lambda^{10} \right). \quad (7.28)$$

As is seen from (7.27), at $c \rightarrow \infty$ the contribution of linear magnetic field to the spectrum really vanishes and spectrum becomes in essence purely Coulomb one (modulo the rest energy) which corresponds to the lines discussed in Subsection 5.1.

8 Uniqueness of the Confining Solutions

8.1 Uniqueness

Let us consider the question of uniqueness of the confining solutions. The latter ones were defined in Section 1 and in Subsection 5.1 as the spherically symmetric solutions of the Yang-Mills equations (B.3) (at $J = 0$) containing only the components of the $SU(N)$ -field which are Coulomb-like or linear in r . Additionally we impose the Lorentz condition (B.4) on the sought solutions. As was remarked in Appendix B, the latter condition is necessary for quantizing the gauge fields consistently within the framework of perturbation theory (see, e. g. Ref. [7]), so we should impose the given condition. Let us for definiteness consider the case of group $SU(3)$. Under this situation we can take the general ansatz of form

$$A = r^\mu \Gamma dt + A_r dr + A_\vartheta d\vartheta + r^\nu \Delta d\varphi, \quad (8.1)$$

where $A_\vartheta = r^\rho T$ and matrices $\Gamma = \alpha^a \lambda_a$, $\Delta = \beta^a \lambda_a$, $T = \gamma^a \lambda_a$ with arbitrary real constants $\alpha^a, \beta^a, \gamma^a$. It could seem that there is a more general ansatz in the form $A = r^{\mu_a} \alpha^a \lambda_a dt + A_r dr + r^{\rho_a} \gamma^a \lambda_a d\vartheta + r^{\nu_a} \beta^a \lambda_a d\varphi$ but somewhat more complicated considerations than the ones below (see Appendix E) show that all the same we should have $\mu_a = \mu, \nu_a = \nu$ for any a so, within the current Subsection, we at once consider this condition to be fulfilled to avoid unnecessary complications and also we put $\rho_a = \rho$ at any a for simplicity though it is not obligatory (see Appendix E).

For the ansatz (8.1) the Lorentz condition (B.4) takes the form

$$\partial_r(r^2 \sin \vartheta g^{rr} A_r) + \partial_\vartheta(r^2 \sin \vartheta g^{\vartheta\vartheta} A_\vartheta) = 0$$

which can be rewritten as

$$\partial_\vartheta(\sin \vartheta A_\vartheta) + \sin \vartheta \partial_r(r^2 A_r) = 0, \quad (8.2)$$

wherefrom it follows $\cot \vartheta r^\lambda T + \partial_r(r^2 A_r) = 0$ while the latter entails

$$A_r = \frac{C}{r^2} - \frac{\cot \vartheta r^{\rho-1} T}{\rho + 1} \quad (8.3)$$

with a constant matrix C belonging to SU(3)-Lie algebra. Then we can see that it should put $C = T = 0$ or else A_r will not be spherically symmetric and the confining one where only the powers of r equal to ± 1 are admissible. As a result we come to the conclusion that one should put $A_r = A_\vartheta = 0$ in (8.1). After this we have $F = dA + gA \wedge A = -\mu r^{\mu-1} \Gamma dt \wedge dr + \nu r^{\nu-1} \Delta dr \wedge d\varphi + gr^{\mu+\nu} [\Gamma, \Delta] dt \wedge d\varphi$ which entails (with the help of (A.6))

$$*F = \mu r^{\mu+1} \sin \vartheta \Gamma d\vartheta \wedge d\varphi - \frac{\nu r^{\nu-1} \Delta}{\sin \vartheta} dt \wedge d\vartheta - \frac{gr^{\mu+\nu}}{\sin \vartheta} [\Gamma, \Delta] dr \wedge d\vartheta$$

and the Yang-Mills equations (B.3) (at $J = 0$) turn into

$$\begin{aligned} \mu(\mu + 1)r^{\mu+1} \sin^2 \vartheta \Gamma &= g^2 r^{\mu+2\nu} [\Delta, [\Gamma, \Delta]], \\ \nu(\nu - 1)r^{\nu-2} \Delta &= g^2 r^{2\mu+\nu} [\Gamma, [\Gamma, \Delta]]. \end{aligned} \quad (8.4)$$

It is now not complicated to enumerate possibilities for obtaining the confining solutions in accordance with (8.4), where we should put $\mu = -1, \nu = 1$.

1. $\Gamma=0$ or $\Delta=0$. This situation does obviously not correspond to a confining solution
2. $\Gamma = C\Delta$ with some constant C . This case conforms to that all the parameters α^a describing electric colour Coulomb field and the ones β^a for linear magnetic colour field are proportional – the situation is not quite clear from physical point of view

3. Matrices Γ, Δ are not equal to zero simultaneously and both matrices belong to Cartan subalgebra of SU(3)-Lie algebra. The parameters α^a, β^a of electric and magnetic colour fields are not connected and arbitrary, i. e. they should be chosen from experimental data. The given situation is the most adequate to the physics in question and the corresponding confining solution is in essence the same which has been obtained in Section 2 in the form (2.13)–(2.14).

One can somewhat generalize the starting ansatz (8.1) taking it in the form $A = (r^\mu \Gamma + A')dt + (r^\nu \Delta + B')d\varphi$ with matrices $A' = A^a \lambda_a$, $B' = B^a \lambda_a$ and constants A^a, B^a . Then considerations along the same above lines draw the conclusion that the nontrivial confining solution is described by Γ, Δ, A', B' belonging to Cartan subalgebra which we could already see in the solutions (2.13)–(2.14).

Clearly, the obtained results may be extended over all SU(N)-groups and even over all semisimple compact Lie groups since for them the corresponding Lie algebras possess just the only Cartan subalgebra. Also we can talk about the compact non-semisimple groups, for example, U(N). In the latter case additionally to Cartan subalgebra we have centrum consisting from the matrices of the form αI_N with arbitrary constant α . Really we have obtained the confining solutions for U(1)-group in Subsection 2.2. The most relevant physical cases are of course U(1)- and SU(3)-ones (QED and QCD), therefore we shall not consider further generalizations of the results obtained. It should also be noted that the nontrivial confining solutions obtained exist at any gauge coupling constant g , i. e. they are essentially *nonperturbative* ones. At last, as we have seen in Sections 5, 6, there exist the nontrivial confining solutions containing the Dirac-like monopole parts. The latter are, however, not the spherically symmetric ones and we shall here not dwell upon uniqueness for such solutions.

8.1.1 Remark Concerning the Wilson Confinement Criterion

In our recent paper [31] it was shown that the confining solutions under consideration satisfy the so-called Wilson confinement criterion formulated as far back as in Ref. [32] (see also Ref. [2]). During a long time up to now this criterion remains the dominant one when building one or another approach to the confinement problem. As far as is known to us, however, so far no explicit solutions of the SU(N)-Yang-Mills equations obeying the given criterion have been found. So, taking into account uniqueness of the confining solutions under discussion, perhaps there exist no other solutions meeting the mentioned criterion.

8.2 Nonrelativistic Confining Potentials

As has been mentioned in Subsection 4.1, in meson spectroscopy (see, e.g., Refs. [14, 33] and references therein) one often uses the nonrelativistic confining potentials. Those confining potentials between quarks here are usually modelled in the form $a/r + br$ with some constants a and b . It is clear, however, that from the QCD point of view the interaction between quarks should be described by the whole SU(3)-field $A_\mu = A_\mu^a T_a$, genuinely

relativistic object, the nonrelativistic potential being only some component of A_t^a surviving in the nonrelativistic limit when the light velocity $c \rightarrow \infty$. Let us explore whether such potentials may be the solutions of the Maxwell or SU(3)-Yang-Mills equations. Though this can be easily derived from the results obtained in Subsection 8.1 let us consider the given situation directly in view of its physical importance.

8.2.1 Maxwell Equations

In the case of Maxwell equations (B.5) (at $J = 0$) the ansatz $A = A_t dt = (a/r + br)dt$ yields $F = dA = (a/r^2 - b)dt \wedge dr$, $*F = \sin \vartheta (br^2 - a)d\vartheta \wedge d\varphi$ and the relation $d * F = 2br \sin \vartheta dr \wedge d\vartheta \wedge d\varphi = 0$ entails $b \equiv 0$.

8.2.2 SU(3)-Yang-Mills Equations

We use the ansatz

$$A = A_t^a \lambda_a dt = (A'/r + B'r)dt \quad (8.5)$$

with some constant matrices $A' = \alpha^a \lambda_a$, $B' = \beta^a \lambda_a$. Then $A \wedge A = 0$, $F = dA + gA \wedge A = dA = (A'/r^2 - B')dt \wedge dr$, $*F = \sin \vartheta (B'r^2 - A')d\vartheta \wedge d\varphi$, $d * F = 2B'r \sin \vartheta dr \wedge d\vartheta \wedge d\varphi$, $*F \wedge A - A \wedge *F = -2[A', B']r dt \wedge d\vartheta \wedge d\varphi$. Under the circumstances the Yang-Mills equations (B.3) (at $J = 0$) are tantamount to the conditions $d * F = 0$, $*F \wedge A - A \wedge *F = 0$. The former entails $B' = 0$, then the latter is fulfilled at any A' and we can see that the Coulomb-like field $A = (A'/r)dt$ is a solution of the Yang-Mills equations (B.3) (at $J = 0$) with arbitrary constant matrix A' which actually has been obtained in Subsection 8.1. In principle the ansatz (8.5) might be a solution of (B.3) with the source of the form

$$J = 2B'r \sin \vartheta dr \wedge d\vartheta \wedge d\varphi + 2g[A', B']r dt \wedge d\vartheta \wedge d\varphi =$$

$$*j = *(j_\mu^a \lambda_a dx^\mu) = * \left(\frac{2B'}{r} dt + g \frac{2[A', B']}{r \sin \vartheta} dr \right), \quad (8.6)$$

but $\text{div}(j) \neq 0$ and this is not consistent with the only source (5.6) derived from the QCD lagrangian (5.1). We can avoid this difficulty putting matrices A' , B' are not equal to zero simultaneously and both matrices belong to Cartan subalgebra of SU(3)-Lie algebra. Then $B' = \beta_3 \lambda_3 + \beta_8 \lambda_8$ and for consistency with the only admissible source of (5.6) we should require source of (5.6) to be equal to one of (8.6) which entails

$$g\bar{\Psi}(\gamma_\mu \otimes I_3) \lambda^a \Psi \lambda_a dx^\mu = \frac{2(\beta^3 \lambda_3 + \beta^8 \lambda_8)}{r} dt,$$

wherefrom one can conclude that

$$g\bar{\Psi}(\gamma_t \otimes I_3) \lambda^a \Psi = 0, a \neq 3, 8, g\bar{\Psi}(\gamma_t \otimes I_3) \lambda^3 \Psi = \frac{2\beta^3}{r},$$

$$g\bar{\Psi}(\gamma_t \otimes I_3) \lambda^8 \Psi = \frac{2\beta^8}{r}, g\bar{\Psi}(\gamma_\mu \otimes I_3) \lambda^a \Psi = 0, a = 1, \dots, 8, \mu \neq t, \quad (8.7)$$

which can obviously be satisfied only at $\beta^3 \sim \beta^8 \sim \Psi \rightarrow 0$ at each point of Minkowski spacetime, i. e., really matrix $B' = 0$ again. All the above can easily be generalized to any $N > 1$.

As a result, the potentials employed in nonrelativistic approaches do not obey the Maxwell or Yang-Mills equations. The latter ones are essentially relativistic and, as we have seen, the components linear in r of the whole A_μ are different from A_t and related with magnetic (colour) field vanishing in the nonrelativistic limit.

8.3 Remark about Search for Nonrelativistic Confining Potentials

The above results make us cast a new glance at search of many years for nonrelativistic potentials modelling the confinement. Many efforts were devoted to the latter topic, for example, within the framework of lattice gauge theories or potential approach (see, e.g., Refs. [34] and references therein). It should be noted, however, that almost in all literature on this direction one does not bring up a question: whether such potentials could (or should) satisfy the Yang-Mills equations? As is clear from the above the answer is negative. That is why the mentioned approaches seem to be inconsistent.

9 Application to the Charmonium Spectrum

As we have emphasized in Subsection 5.1, all considerations in the given paper develop with the aim of further physical applications or else the results obtained could be only of academic interest. Though there are no obstacles to apply the results (in the most physically interesting case $N = 3$) to any meson, up to the moment of writing this paper all applications were concentrated on quarkonia (charmonium and bottomonium) and the results are contained in Refs. [5, 6, 35]. Referring for more details to those papers, we shall here outline only the main conclusions drawn from the mentioned papers which confirm the physical picture underlying considerations of the given paper and warrant the linear confinement scenario described in Subsection 4.2.

9.1 Relativistic Spectrum of Charmonium

We can use (7.17)–(7.19) with various combinations of signes (\pm) before second summand in numerators of those formulas. In Refs. [5, 6, 35] due to some reasons (inessential now) the combination $(+ + -)$ was employed and besides the replacement $\omega_2 \rightarrow \omega_3$ was made. Let us rewrite, e. g., the results out of Ref. [35] about spectrum according to (7.17)–(7.19) but keeping the mentioned combination $(+ + -)$. Then numerical results for constants parametrizing the charmonium spectrum are shown in Table 1.

One can note that the obtained mass parameter μ_0 is consistent with the present-day experimental limits [36] where the current mass of c -quark ($2\mu_0$) is accepted between 1.1 GeV and 1.4 GeV. As for the gauge coupling constant g then its value has been chosen in accordance with many recent considerations [37], wherefrom one can conclude that the

Table 1: Gauge coupling constant, mass parameter μ_0 and parameters of the confining SU(3)-gluonic field for charmonium.

g	μ_0 (GeV)	a_1	a_2	b_1 (GeV)	b_2 (GeV)	B_1	B_2
0.46900	0.62500	2.21104	-0.751317	20.2395	-12.6317	6.89659	6.89659

strong coupling constant $\alpha_s = g^2$ is of order $0.22 \approx 0.469^2$ at the scale of the c -quark current mass. At last, as to parameters $A_{1,2}$ of solution (2.13), then they only shift the origin of count for the corresponding energies and we can consider $A_1 = A_2 = 0$, as was mentioned in Subsections 6.2, 6.4.

With the constants of Table 1 the present-day levels of charmonium spectrum were calculated with the help of (7.17)–(7.19) so Table 2 contains experimental values of these levels (from Ref. [36]) and our theoretical ones computed according to the shown combinations (we use the notations of levels from Ref. [36]).

Table 2: Theoretical and experimental charmonium levels.

State	Theoret. energy $\varepsilon_j = \sum_{k=1}^3 \omega_k$ (GeV)	Experm. value (GeV)
$\eta_c(1S)$	$\varepsilon_1 = \omega_1(0, 0, -1) + \omega_2(0, 0, -1) + \omega_3(0, 0, -1) = 2.979597$	2.979600
$J/\psi(1S)$	$\varepsilon_2 = \omega_1(0, 0, -1) + \omega_2(0, 0, -1) + \omega_3(0, 0, 1) = 3.096913$	3.096916
$\chi_{c0}(1P)$	$\varepsilon_3 = \omega_1(0, 0, -1) + \omega_2(0, 0, 1) + \omega_3(0, 0, -1) = 3.415186$	3.415190
$\chi_{c1}(1P)$	$\varepsilon_4 = \omega_1(0, 0, 1) + \omega_2(0, 1, -1) + \omega_3(2, 0, 1) = 3.505304$	3.510590
$h_c(1P)$	$\varepsilon_5 = \omega_1(0, 0, -1) + \omega_2(0, 0, 1) + \omega_3(0, 0, 1) = 3.532503$	3.526210
$\chi_{c2}(1P)$	$\varepsilon_6 = \omega_1(0, 1, -1) + \omega_2(1, 1, -1) + \omega_3(1, 1, -1) = 3.553097$	3.556260
$\eta_c(2S)$	$\varepsilon_7 = \omega_1(0, 0, 1) + \omega_2(0, 1, -1) + \omega_3(1, 0, -1) = 3.671608$	3.65400
$\psi(2S)$	$\varepsilon_8 = \omega_1(0, 1, -1) + \omega_2(1, 1, -1) + \omega_3(2, 1, 1) = 3.674025$	3.685093
$\psi(3770)$	$\varepsilon_9 = \omega_1(0, 0, 1) + \omega_2(0, 0, 1) + \omega_3(2, 0, -1) = 3.775598$	3.770000
$\psi(3836)$	$\varepsilon_{10} = \omega_1(0, 1, -1) + \omega_2(0, 1, 1) + \omega_3(0, 0, 1) = 3.833640$	3.836000
$X(3872)$	$\varepsilon_{11} = \omega_1(0, 1, -1) + \omega_2(0, 1, 1) + \omega_3(0, 1, 1) = 3.871672$	3.872000
$\psi(4040)$	$\varepsilon_{12} = \omega_1(0, 0, 1) + \omega_2(0, 1, -1) + \omega_3(1, 1, 1) = 4.042660$	4.040000
$\psi(4160)$	$\varepsilon_{13} = \omega_1(0, 0, -1) + \omega_2(0, 1, 1) + \omega_3(0, 0, -1) = 4.153765$	4.159000
$\psi(4415)$	$\varepsilon_{14} = \omega_1(0, 0, -1) + \omega_2(1, 0, -1) + \omega_3(2, 1, 1) = 4.409260$	4.415000

9.2 Nonrelativistic Limit

One can be interested in estimating the contribution of relativistic effects, i. e. those connected with magnetic colour field linear in r . This has been done in Refs. [5, 6] with the help of relations (7.27)–(7.28). The contribution of relativistic effects can amount to tens per cent and they cannot be considered small. Moreover, the more excited the state of charmonium the worse the nonrelativistic approximation. The physical reason of it is quite clear. Really, we have seen in the nonrelativistic limit (see the relations [7.27]–(7.28)] that the parameters $b_{1,2}$, $B_{1,2}$ of the linear interaction between quarks vanish under this limit and the nonrelativistic spectrum is independent of them and is practically getting the pure Coulomb one. As a consequence, the picture of linear confinement for quarks should be considered an essentially relativistic one while the nonrelativistic limit is very crude approximation. In fact, as follows from exact solutions of SU(3)-Yang–Mills equations of (2.13), the linear interaction between quarks is connected with magnetic colour field that dies out in the nonrelativistic limit, *i.e.* for static quarks. Only for the moving rapidly enough quarks the above field will appear and generate linear confinement between them. So the spectrum will depend on both the static Coulomb electric colour field and the dynamical magnetic colour field responsible for the linear confinement for quarks which is just confirmed by the relations (7.17)–(7.19). In our case, the interaction effect with the magnetic colour field is taken into consideration from the very outset which just reflects the linear confinement at large distances.

9.3 Electromagnetic Transitions

We can specify the obtained above charmonium spectrum. The fact is that the relations (7.17)–(7.19) permit various parametrizations of the charmonium spectrum (see Refs. [5, 6]) and therefore it should impose further conditions to fix a certain parametrization among several possible ones. For example, one can compute widths of electromagnetic transitions among levels of charmonium, in particular for transitions $J/\psi(1S) \rightarrow \eta_c(1S) + \gamma$ and $\chi_{c0}(1P) \rightarrow J/\psi(1S) + \gamma$. In Ref. [35] the widths of the latter transitions have been calculated in dipole approximation that allowed one to use the corresponding wave functions described in Subsection 7.3. Results of computation supplies us with an additional justification for the choice of parameters of the SU(3)-confining gluon field adduced in Table 1 and allows us to conclude that dipole approximation is not enough for the second transition of the ones under discussion. The question now is what gluon concentrations are in the mentioned SU(3)-confining gluon field.

9.4 Estimates of Gluon Concentrations and Magnetic Colour Field Sstrength

To obtain necessary estimates we shall use an analogy with classical electrodynamics where is well known (see e. g. [10]) that the notion of classical electromagnetic field (a photon condensate) generated by a charged particle is applicable only at distances \gg the Compton wavelength $\lambda_c = 1/m$ for the given particle. If denoting λ_B the de Broglie wavelength

of the particle then $\lambda_B = 1/p$ with the relativistic impulse $p = mv/\sqrt{1-v^2}$ while v is the velocity of the particle (as a result, $\lambda_c = \lambda_B$ at $v = 1/\sqrt{2}$) so one can rewrite $\lambda_B = \lambda_c\sqrt{1-v^2}/v$ and it is clear that $\lambda_B \rightarrow 0$ when $v \rightarrow 1$ (ultrarelativistic case), i. e., the particle becomes more and more point-like one. Accordingly, one can conclude that in the latter case the notion of classical electromagnetic field generated by a charged ultrarelativistic particle is applicable at distances $\gg \lambda_B$. Under the circumstances, if the ultrarelativistic charged particle accomplishes its motion within the region with characteristic size of order r_0 then in the given region the electromagnetic field generated by the particle may be considered as classical one at $r_0 \gg \lambda_B$. For example, in the case of positronium we have $r_0 \sim 2a_0$, where $a_0 \approx 5.29 \cdot 10^{-11}$ m is the Bohr radius, so $r_0 \gg \lambda_e \approx 3.86 \cdot 10^{-13}$ m, the electron Compton wavelength, i. e., the electric Coulomb interaction between electron and positron in positronium can be considered classical electromagnetic field.

Passing on to QCD, gluons and quarkonia, it should be noted that quarks in quarkonia accomplish a finite motion within a region of order $1 \text{ fm} = 10^{-15}$ m. Then, as is seen from the radial parts of the wave functions described in Subsection 7.3, the quantity $1/\beta_j$ permits to be considered a characteristic size of the j -th colour component of the given quarkonium state and, consequently, we can take the magnitude

$$r_0 = \frac{1}{3} \sum_{j=1}^3 \frac{1}{\beta_j} \quad (9.1)$$

for a characteristic size of the whole quarkonium state and, in line with the above, we should consider the confining SU(3)-gluonic Yang-Mills field of (2.13) or (2.14) to be classical one when $r_0 \gg \lambda_B$, the de Broglie wavelength of the corresponding quarks forming quarkonium.

On the other hand, a classical electromagnetic field (photon condensate) conforms to the large photon concentrations for every frequency presented in the field [23]. Then in QCD we should require the large gluon concentrations in the given classical gluonic field (gluon condensate). For the necessary estimates we shall employ the T_{00} -component of the energy-momentum tensor for a SU(N)-Yang-Mills field

$$T_{\mu\nu} = \frac{1}{4\pi} \left(-F_{\mu\alpha}^a F_{\nu\beta}^a g^{\alpha\beta} + \frac{1}{4} F_{\beta\gamma}^a F_{\alpha\delta}^a g^{\alpha\beta} g^{\gamma\delta} g_{\mu\nu} \right). \quad (9.1')$$

To estimate the given concentrations we can employ T_{00} -component (volumetric energy density) of the energy-momentum tensor of (9.1') and, taking the quantity $\omega = \Gamma$, the whole decay width of the quarkonium state, for the characteristic frequency we obtain the sought characteristic concentration n in the form

$$n = \frac{T_{00}}{\Gamma}. \quad (9.2)$$

It is not complicated to obtain the curvature matrix (field strength) corresponding to the solution (2.13) or (2.14)

$$F = F_{\mu\nu}^a \lambda_a dx^\mu \wedge dx^\nu = -\partial_r(A_t^a \lambda_a) dt \wedge dr + \partial_r(A_\varphi^a \lambda_a) dr \wedge d\varphi, \quad (9.3)$$

which entails the only nonzero components

$$F_{tr}^3 = \frac{a_1 - a_2}{2r^2}, F_{tr}^8 = \frac{(a_1 + a_2)\sqrt{3}}{2r^2}, F_{r\varphi}^3 = \frac{b_1 - b_2}{2}, F_{r\varphi}^8 = \frac{(b_1 + b_2)\sqrt{3}}{2} \quad (9.4)$$

and, in its turn,

$$T_{00} \equiv T_{tt} = \frac{1}{4\pi} \left\{ \frac{3}{4} \left[(F_{tr}^3)^2 + (F_{tr}^8)^2 \right] + \frac{1}{4r^2 \sin^2 \vartheta} \left[(F_{r\varphi}^3)^2 + (F_{r\varphi}^8)^2 \right] \right\} =$$

$$\frac{3}{16\pi} \left(\frac{a_1^2 + a_1 a_2 + a_2^2}{r^4} + \frac{b_1^2 + b_1 b_2 + b_2^2}{3r^2 \sin^2 \vartheta} \right), \quad (9.5)$$

so, further putting $\sin^2 \vartheta = 1/3$ for simplicity, we can rewrite (9.5) in the form

$$T_{00} = T_{00}^{\text{coul}} + T_{00}^{\text{lin}} \quad (9.6)$$

conforming to the contributions from the Coulomb and linear parts of the solutions (2.13) or (2.14). The latter gives the corresponding split of n from (9.2)

$$n = n_{\text{coul}} + n_{\text{lin}}. \quad (9.7)$$

Using the Hodge star operator in 3-dimensional euclidean space [where $ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu \equiv dr^2 + r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$, $*(dr \wedge d\vartheta) = \sin \vartheta d\varphi$, $*(dr \wedge d\varphi) = -d\vartheta/\sin \vartheta$, $*(d\vartheta \wedge d\varphi) = dr/(r^2 \sin \vartheta)$], we can confront components $F_{r\varphi}^3, F_{r\varphi}^8$ with 3-dimensional 1-forms of the magnetic colour field

$$\mathbf{H}^3 = -\frac{b_1 - b_2}{2 \sin \vartheta} d\vartheta, \mathbf{H}^8 = -\frac{(b_1 + b_2)\sqrt{3}}{2 \sin \vartheta} d\vartheta$$

that are modulo equal to $\sqrt{g^{\mu\nu} \mathbf{H}_\mu^{3,8} \mathbf{H}_\nu^{3,8}}$ or

$$H^3 = \frac{|b_1 - b_2|}{2r \sin \vartheta}, H^8 = \frac{|b_1 + b_2|\sqrt{3}}{2r \sin \vartheta}, \quad (9.8)$$

where we also consider $\sin^2 \vartheta = 1/3$ for simplicity.

For comparison we shall also estimate the photon concentration in the ground state of the positronium. As is known historically [38], the analogy between the latter system and quarkonia played the important part when building the quarkonia models. For positronium we have the electromagnetic Coulomb interaction $A = A_t dt = (e/r)dt$ which entails $F = F_{tr} dt \wedge dr = (e/r^2)dt \wedge dr$ and

$$T_{00} \equiv T_{tt} = \frac{1}{4\pi} (F_{tr})^2 = \frac{\alpha_{em}}{4\pi r^4} \quad (9.9)$$

with $\alpha_{em} = e^2 = 1/137.0359895$.

9.5 Numerical Results and Concluding Remarks

When computing for the ground state of charmonium we used its present-day whole decay width $\Gamma = 17.3$ MeV [36], while the calculation r_0 of (9.1) gives $r_0 = r_1$ (see Table 3).

In the positronium case we employed the widths $\Gamma_0 = 1/\tau_0$ (parapositronium) and $\Gamma_1 = 1/\tau_1$ (orthopositronium), respectively, with the life times $\tau_0 = 1.252 \cdot 10^{-10}$ s, $\tau_1 = 1.377 \cdot 10^{-7}$ s [38] while $r_0 = 2a_0$ with the Bohr radius $a_0 = 0.529177249 \cdot 10^5$ fm [36].

Tables 3, 4 contain the numerical results for both the cases, where, when calculating, we applied the relations $1 \text{ GeV}^{-1} \approx 0.21030893 \text{ fm}$, $1 \text{ T} \approx 0.692508 \times 10^{-15} \text{ GeV}^2$.

Table 3: Gluon concentrations and magnetic colour field strengths in the ground state of charmonium.

$\eta_c(1S): \quad r_1 = 0.0399766 \text{ fm}$					
r (fm)	n_{coul} (m^{-3})	n_{lin} (m^{-3})	n (m^{-3})	H^3 (T)	H^8 (T)
$0.1r_1$	$0.550649 \cdot 10^{58}$	$0.727630 \cdot 10^{55}$	$0.551377 \cdot 10^{58}$	$0.216259 \cdot 10^{19}$	$0.866919 \cdot 10^{18}$
r_1	$0.550649 \cdot 10^{54}$	$0.727630 \cdot 10^{53}$	$0.623412 \cdot 10^{54}$	$0.216259 \cdot 10^{18}$	$0.866919 \cdot 10^{17}$
$10r_1$	$0.550649 \cdot 10^{50}$	$0.727630 \cdot 10^{51}$	$0.782695 \cdot 10^{51}$	$0.216259 \cdot 10^{17}$	$0.866919 \cdot 10^{16}$
1.0	$0.140637 \cdot 10^{49}$	$0.116285 \cdot 10^{51}$	$0.117691 \cdot 10^{51}$	$0.864530 \cdot 10^{16}$	$0.346565 \cdot 10^{16}$
a_0	$0.179347 \cdot 10^{30}$	$0.415260 \cdot 10^{41}$	$0.415260 \cdot 10^{41}$	$0.163372 \cdot 10^{12}$	$0.654912 \cdot 10^{11}$

Table 4: Photon concentrations in the ground state of positronium.

$r_0 = 2a_0 = 2 \cdot 0.529177249 \cdot 10^5 \text{ fm}$		
	Parapositronium	Orthopositronium
r (fm)	n_{para} (m^{-3})	n_{ortho} (m^{-3})
$0.01r_0$	$0.888025 \cdot 10^{46}$	$0.976685 \cdot 10^{49}$
$0.1r_0$	$0.888025 \cdot 10^{42}$	$0.976685 \cdot 10^{45}$
r_0	$0.888025 \cdot 10^{38}$	$0.976685 \cdot 10^{41}$
$2r_0$	$0.555015 \cdot 10^{37}$	$0.610428 \cdot 10^{40}$

Then, as is seen from Tables 3, 4, qualitative behaviour of both the concentrations is similar. At the characteristic scales of each system the concentrations are large and the corresponding fields (electric and magnetic colour ones or electric Coulomb one) can be considered classical ones. For charmonium the part n_{coul} of gluon concentration n connected with the Coulomb electric colour field is decreasing faster than n_{lin} , the part of n related to the linear magnetic colour field, and at large distances n_{lin} becomes dominant.

Under the circumstances, as has been said above, we can estimate the quark velocities in the charmonium state under discussion from the condition

$$v = \frac{1}{\sqrt{1 + \left(\frac{\lambda_B}{\lambda_q}\right)^2}} \quad (9.10)$$

with the c -quark Compton wavelength $\lambda_q = 1/(2\mu_0) \approx 0.168247$ fm so, taking the de Broglie wavelength $\lambda_B = 0.1r_1$ with r_1 from Table 3, we obtain $v \approx 0.999718$, i. e., the quarks in charmonium should be considered the ultrarelativistic point-like particles. This additionally confirms the conclusion of Refs. [5, 6] that the relativistic effects are extremely important for the confinement mechanism. As a result, the confinement scenario described in Subsection 4.2 may really occur. At last, we can see that strength of magnetic colour field responsible for linear confinement reaches huge values of order 10^{17} – 10^{18} T. For comparison one should notice that the most strong magnetic fields known at present have been discovered in magnetic neutron stars, pulsars (see, e. g., Ref. [39]) where the corresponding strengths can be of order 10^9 – 10^{10} T. So the characteristic feature of confinement is really the very strong magnetic colour field between quarks which we have emphasized in Subsection 5.1. In a certain sense the essence of confinement can be said to be just in enormous gluon concentrations and magnetic colour field strengths in space around quarks.

10 Conclusion

Throughout the paper we moved step-by-step forward in analysing the Dirac-Yang-Mills system of equations (5.4)–(5.5) derived from the QCD lagrangian. The aim we pursued was to obtain a scenario for linear confinement of quarks, at any rate, in mesons and quarkonia. It seems to us we succeeded in finding the suitable quantitative description for the given phenomenon. As we could see, crucial step here consisted in studying exact solutions of the SU(3)-Yang-Mills equations modelling confinement and the corresponding modulo square integrable solutions of the Dirac equation. Techniques of finding the mentioned solutions were based to a great degree on those borrowed from black hole theory. In this respect our approach is conventional for physics whose whole history shows that very often the methods developed for solving some problems proved to be extremely useful for analysing a number of other ones in a perfectly different region of physics.

At the end of our considerations let us summarize the main features of the confinement mechanism developed in the given paper.

1. The whole approach is based on the exact solutions of the Yang-Mills equations and on the corresponding modulo square integrable solutions of Dirac equation. The solutions in question are essentially unique ones: for the confining solutions of the SU(3)-Yang-Mills equations that was shown in Section 8 whereas uniqueness for the conforming solutions of Dirac equation (5.4) follows from the ansatz 7.1'. Namely, the mentioned ansatz leads to (7.3) which is the confluent hypergeometric equation

and (as is known from literature on special functions, see, e. g., Ref. [29]) the latter just possesses the only suitable solutions for us in the form of Laguerre polynomials which in essence determines the spectrum of relativistic bound states [see relations (7.7)–(7.8)]. Consequently, we found the only compatible solutions of the system (5.4)–(5.5) which can have pretensions of describing the confinement mechanism. Another matter that lagrangian QCD probably allows one to develop some other approaches [not based on compatible solutions of (5.4)–(5.5)] to confinement but our one seems to be the most natural since it is practically identical to the standard approach of quantum mechanics and QED to description of bound states in external electromagnetic fields. In other words, this approach should have been one of the very first approaches as soon as the QCD lagrangian was written. Historically, however, this way was rejected due to incomprehensible reasons.

2. Two main physical reasons for linear confinement in the mechanism under discussion are the following ones. The first one is that gluon exchange between quarks is realized with the propagator different from the photon one and existence of such a propagator is direct sequence of the unique confining solutions of the Yang-Mills equations. The second reason is that, owing to the structure of mentioned propagator, gluon condensate (a classical gluon field) between quarks mainly consists of soft gluons (see Subsection 4.1) but, because of that any gluon also emits gluons, the corresponding gluon concentrations rapidly become huge and form the linear confining magnetic colour field of enormous strengths which leads to confinement of quarks. Under the circumstances physically nonlinearity of the Yang-Mills equations effectively vanishes so the latter possess only the unique confining solutions of the abelian-like form (with the values in Cartan subalgebra) that describe the gluon condensate under consideration. Moreover, since the overwhelming majority of gluons are soft they cannot leave hadron (meson) until some gluon obtains additional energy (due to an external reason) to rush out. The latter seems to be observable just in the so-called 3-jets events (for more details see, e. g., Ref. [38]). So we deal with confinement of gluons as well.
3. The approach under discussion equips us with the explicit wave functions that is practically unreachable in other approaches, for example, within framework of lattice gauge theories or potential approach. Namely, for each meson there exists its own set of real constants a_j, A_j, b_j, B_j parametrizing the mentioned confining gluon field (the gluon condensate) and the corresponding wave functions while the latter ones also depend on μ_0 , the reduced mass of the current masses of quarks forming meson. It is clear that constants $a_j, A_j, b_j, B_j, \mu_0$ should be extracted from experimental data. This circumstance gives possibilities for direct physical modelling of internal structure for any meson and for checking such relativistic models numerically.
4. Finally, there is also an interesting possibility of indirect experimental verification of the confinement mechanism under discussion. Really solutions (2.6) point out the confinement phase could be in electrodynamics as well. Though there exist no

elementary charged particles generating a constant magnetic field linear in r , the distance from particle, after all, if it could generate this electromagnetic field configuration in laboratory then one might study motion of trial charged particles in that field. The confining properties of the mentioned field should be displayed at classical level too but the exact behaviour of particles in this field requires certain analysis of the corresponding classical equations of motion.

At this point we would like to mark the end of the given paper on pinning our hopes on studying confinement with subsequent development of the results obtained here, in the first place, by further applications to concrete mesons.

11 Appendix A: Hodge Star Operator $*$ on Minkowski Spacetime in Spherical Coordinates

Let M is a smooth manifold of dimension n so we denote an algebra of smooth functions on M as $F(M)$. In a standard way the spaces of smooth differential p -forms $\Lambda^p(M)$ ($0 \leq p \leq n$) are defined over M as modules over $F(M)$ (see, e. g. Refs. [20]). If a (pseudo)riemannian metric $G = ds^2 = g_{\mu\nu}dx^\mu \otimes dx^\nu$ is given on M in local coordinates $x = (x^\mu)$ then G can naturally be continued on spaces $\Lambda^p(M)$ by relation

$$G(\alpha, \beta) = \det\{G(\alpha_i, \beta_j)\} \quad (A.1)$$

for $\alpha = \alpha_1 \wedge \alpha_2 \dots \wedge \alpha_p$, $\beta = \beta_1 \wedge \beta_2 \dots \wedge \beta_p$, where for 1-forms $\alpha_i = \alpha_\mu^{(i)} dx^\mu$, $\beta_j = \beta_\nu^{(j)} dx^\nu$ we have $G(\alpha_i, \beta_j) = g^{\mu\nu} \alpha_\mu^{(i)} \beta_\nu^{(j)}$ with the Cartan's wedge (external) product \wedge . Under the circumstances the Hodge star operator $*$: $\Lambda^p(M) \rightarrow \Lambda^{n-p}(M)$ is defined for any $\alpha \in \Lambda^p(M)$ by

$$\alpha \wedge (*\alpha) = G(\alpha, \alpha) \omega_g \quad (A.2)$$

with the volume n -form $\omega_g = \sqrt{|\det(g_{\mu\nu})|} dx^1 \wedge \dots \wedge dx^n$. For example, for 2-forms $F = F_{\mu\nu} dx^\mu \wedge dx^\nu$ we have

$$F \wedge *F = (g^{\mu\alpha} g^{\nu\beta} - g^{\mu\beta} g^{\nu\alpha}) F_{\mu\nu} F_{\alpha\beta} \sqrt{\delta} dx^1 \wedge dx^2 \dots \wedge dx^n, \mu < \nu, \alpha < \beta \quad (A.3)$$

with $\delta = |\det(g_{\mu\nu})|$. If s is the number of (-1) in a canonical presentation of quadratic form G then two most important properties of $*$ for us are

$$*^2 = (-1)^{p(n-p)+s}, \quad (A.4)$$

$$*(f_1 \alpha_1 + f_2 \alpha_2) = f_1 (*\alpha_1) + f_2 (*\alpha_2) \quad (A.5)$$

for any $f_1, f_2 \in F(M)$, $\alpha_1, \alpha_2 \in \Lambda^p(M)$, i. e., $*$ is a $F(M)$ -linear operator. Due to (A.5) for description of $*$ -action in local coordinates it is enough to specify $*$ -action on the basis elements of $\Lambda^p(M)$, i. e. on the forms $dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p}$ with $i_1 < i_2 < \dots < i_p$ whose number is equal to $C_n^p = \frac{n!}{(n-p)!p!}$.

The most important case of M in the given paper is the Minkowski spacetime with local coordinates t, r, ϑ, φ , where r, ϑ, φ stand for spherical coordinates on spatial part of M . The metric is given by (1.1) and we shall obtain the $*$ -action on the basis differential forms according to (A.2)

$$\begin{aligned}
*dt &= r^2 \sin \vartheta dr \wedge d\vartheta \wedge d\varphi, \quad *dr = r^2 \sin \vartheta dt \wedge d\vartheta \wedge d\varphi, \\
*d\vartheta &= -r \sin \vartheta dt \wedge dr \wedge d\varphi, \quad *d\varphi = r dt \wedge dr \wedge d\vartheta, \\
*(dt \wedge dr) &= -r^2 \sin \vartheta d\vartheta \wedge d\varphi, \quad *(dt \wedge d\vartheta) = \sin \vartheta dr \wedge d\varphi, \\
*(dt \wedge d\varphi) &= -\frac{1}{\sin \vartheta} dr \wedge d\vartheta, \quad *(dr \wedge d\vartheta) = \sin \vartheta dt \wedge d\varphi, \\
*(dr \wedge d\varphi) &= -\frac{1}{\sin \vartheta} dt \wedge d\vartheta, \quad *(d\vartheta \wedge d\varphi) = \frac{1}{r^2 \sin \vartheta} dt \wedge dr, \\
*(dt \wedge dr \wedge d\vartheta) &= \frac{1}{r} d\varphi, \quad *(dt \wedge dr \wedge d\varphi) = -\frac{1}{r \sin \vartheta} d\vartheta, \\
*(dt \wedge d\vartheta \wedge d\varphi) &= \frac{1}{r^2 \sin \vartheta} dr, \quad *(dr \wedge d\vartheta \wedge d\varphi) = \frac{1}{r^2 \sin \vartheta} dt,
\end{aligned} \tag{A.6}$$

so that on 2-forms $*^2 = -1$, as should be in accordance with (A.4).

At last it should be noted that all the above is easily over linearity continued on the matrix-valued differential forms (see, e. g., Ref.[21]), i. e., on the arbitrary linear combinations of forms $a_{i_1 i_2 \dots i_p} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p}$, where coefficients $a_{i_1 i_2 \dots i_p}$ belong to some space of matrices V , for example, a $SU(N)$ -Lie algebra. In the latter case, if T_a are matrices of generators of the $SU(N)$ -Lie algebra in N -dimensional representation, we continue the above scalar product G on the $SU(N)$ -Lie algebra valued 1-forms $A = A_\mu^a T_a dx^\mu$ and $B = B_\nu^b T_b dx^\nu$ by the relation

$$G(A, B) = g^{\mu\nu} A_\mu^a B_\nu^b \text{Tr}(T_a T_b), \tag{A.7}$$

where Tr signifies the trace of a matrix, and on linearity with the help of (A.1), (A.7) can be continued over any $SU(N)$ -Lie algebra valued forms.

12 Appendix B: $SU(N)$ -Lie Algebras and their Cartan Subalgebras

$SU(N)$ -Yang-Mills Fields as Connections in Vector Bundles

The most convenient mathematical treatment of classical $SU(N)$ -Yang-Mills fields is, to our mind, the one describing those fields as connections in vector (or, which is equivalent, in principal) bundles over some manifold M . When M is the Minkowski space the situation is simplified so long as all the bundles over M with topology \mathbb{R}^4 are trivial. Referring for more details, e. g. to Refs. [40], we shall restrict ourselves to a few remarks.

If denoting ξ the standard (trivial) N -dimensional vector bundle over Minkowski space-time we can introduce an $SU(N)$ -connection (a classical $SU(N)$ -Yang-Mills field) in ξ as a $SU(N)$ -Lie algebra valued form (the connection matrix) $A = A_\mu dx^\mu$, $A_\mu = A_\mu^a T_a$ while the matrices T_a form a basis of the Lie algebra of group $SU(N)$ in N -dimensional space (we consider T_a hermitian, as is accepted in physics), $a = 1, \dots, N^2 - 1$. Then the curvature matrix (field strength) for ξ -bundle is $F = dA + gA \wedge A = F_{\mu\nu}^a T_a dx^\mu \wedge dx^\nu$ with the exterior differential d (for example, $d = \partial_t dt + \partial_r dr + \partial_\vartheta d\vartheta + \partial_\varphi d\varphi$ in coordinates t, r, ϑ, φ) while a gauge coupling constant g is introduced from physical considerations. Any smooth function (the gauge transformation) $S : M \rightarrow SU(N)$ gives the so-called trivialization of ξ -bundle and if A_S and F_S are the connection and curvature matrices, respectively, for the given trivialization then they are related to the previous ones by relations (the gauge transformations)

$$A_S = S^{-1}AS + \frac{i}{g}S^{-1}dS, \quad F_S = S^{-1}FS, \quad (B.1)$$

where the factor i is taken in order to an $SU(N)$ -Lie algebra could be chosen from the hermitian matrices.

Mathematically A and F are linked by the Bianchi identity holding true for any connection

$$dF = F \wedge A - A \wedge F, \quad (B.2)$$

and physical considerations impose the Yang-Mills equations

$$d * F = g(*F \wedge A - A \wedge *F) + J, \quad (B.3)$$

where $*$ means the Hodge star operator conforming to a Minkowski metric, for instance, in the form of (1.1), while J is a source, i. e., some differential $SU(N)$ -algebra valued 3-form. An arbitrary $SU(N)$ -connection does not obey the equations (B.3) except for the so-called self-dual fields for those $F = *F$ and the equations (B.2) and (B.3) become the same (at $J = 0, g = 1$). But, as was remarked in Appendix A, in Minkowski spacetime $*^2 = -1$ on 2-forms that entails $F = 0$ for self-dual fields.

Also it should be noted that the solutions of (B.3) are usually believed to obey an additional condition. In the present paper we take the Lorentz condition that can be written in the form $\text{div}(A) = 0$, where the divergence of the Lie algebra valued 1-form $A = A_\mu^a T_a dx^\mu$ is defined by the relation (see, e. g. Refs. [20])

$$\text{div}(A) = \frac{1}{\sqrt{\delta}} \partial_\mu (\sqrt{\delta} g^{\mu\nu} A_\nu). \quad (B.4)$$

It should be emphasized that the Lorentz condition is necessary for quantizing the gauge fields consistently within the framework of perturbation theory (see, e. g. Ref. [7]), so we should impose the given condition.

At last, the case of group $U(1)$ is also very important. We then have the case of classical electromagnetic field (the corresponding Lie algebra consists from real numbers), the

Bianchi identity is converted into the the first pair of Maxwell equations $dF = 0$ while the Yang-Mills equations (B.3) become the second pair of Maxwell equations

$$d * F = J \quad (B.5)$$

with $F = dA$, $A = A_\mu dx^\mu$ and $J = j_\mu * (dx^\mu)$ with a 4-dimensional electromagnetic density current $j = j_\mu dx^\mu$, where $j_\mu = j_\mu(x)$ are some functions on \mathbb{R}^4 .

Let us describe the explicit realizations of $SU(N)$ -Lie algebras by hermitian matrices that are needed for our aims and also indicate the so-called Cartan subalgebras in them which is important in the main part of the paper. By definition, a Cartan subalgebra is a maximal abelian subalgebra in the corresponding Lie algebra, i. e., the commutator of any two matrices of the Cartan subalgebra is equal to zero. Dimension of Cartan subalgebra (as a vector space) for $SU(N)$ -Lie algebra is equal to $N - 1$.

$$N = 2$$

In this case we can take $T_a = \sigma_a$ at $a = 1, 2, 3$, where σ_a are the ordinary Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (B.6)$$

We shall also notice that

$$\text{Tr}(\sigma_k \sigma_j) = \frac{1}{2} \delta_{kj}. \quad (B.7)$$

The Cartan subalgebra is generated by σ_3 .

$$N = 3$$

In the given situation we can take $T_a = \lambda_a$, where λ_a are the Gell-Mann matrices

$$\begin{aligned} \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \end{aligned} \quad (B.8)$$

and

$$\text{Tr}(\lambda_k \lambda_j) = 2 \delta_{kj}. \quad (B.9)$$

The Cartan subalgebra is generated by λ_3 and λ_8 .

$N = 4$

Following Ref. [41] we take (I_2 stands for the unit matrix 2×2)

$$\begin{aligned} T_1 &= \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, T_2 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, T_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \\ T_4 &= \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, T_5 = \begin{pmatrix} 0 & -iI_2 \\ iI_2 & 0 \end{pmatrix}, T_6 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \end{aligned} \quad (B.10)$$

and the products of matrices $T_7 = T_1T_4, T_8 = T_1T_5, T_9 = T_1T_6, T_{10} = T_2T_4, T_{11} = T_2T_5, T_{12} = T_2T_6, T_{13} = T_3T_4, T_{14} = T_3T_5, T_{15} = T_3T_6$ for the rest of the Lie algebra basis so the Cartan subalgebra is generated by T_3, T_6, T_{15} with

$$T_{15} = T_3T_6 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}. \quad (B.11)$$

Then

$$\text{Tr}(T_k T_j) = 4\delta_{kj}. \quad (B.12)$$

13 Appendix C: Eigenspinors of the Euclidean Dirac Operator on \mathbb{S}^2 at $\lambda = \pm 1$

When using the wave functions obtained in Section 7 in any applications (see, e.g., Ref. [35]) one needs the explicit form for eigenspinors of the euclidean Dirac operator \mathcal{D}_0 (see Subsection 6.1). Though it is in general given by the relation (6.6), for applications, as a rule, it is sufficient to restrict oneself to small eigenvalues of \mathcal{D}_0 . Let us write out, therefore, the eigenspinors of the euclidean Dirac operator \mathcal{D}_0 corresponding to the eigenvalues $\lambda = \pm 1$ in explicit form.

If $\lambda = \pm(l+1) = \pm 1$ then $l = 0$ and from (6.6) it follows that $k = l + 1/2 = 1/2$, $|m'| < 1/2$ and we need the functions $P_{\pm 1/2 \pm 1/2}^{1/2}$. But according to Ref. [27] we have $P_{kk}^k = \cos^{2k}(\vartheta/2)$, $P_{k-k}^k = i^{2k} \sin^{2k}(\vartheta/2)$ and, besides, $P_{kk}^k = P_{-k-k}^k$, $P_{-kk}^k = P_{k-k}^k$ that entails the eigenspinors for $\lambda = -1$ in the form

$$\Phi = \frac{C}{2} \begin{pmatrix} \cos \frac{\vartheta}{2} + i \sin \frac{\vartheta}{2} \\ \cos \frac{\vartheta}{2} - i \sin \frac{\vartheta}{2} \end{pmatrix} e^{i\varphi/2}, \Phi = \frac{C}{2} \begin{pmatrix} \cos \frac{\vartheta}{2} + i \sin \frac{\vartheta}{2} \\ -\cos \frac{\vartheta}{2} + i \sin \frac{\vartheta}{2} \end{pmatrix} e^{-i\varphi/2}, \quad (C.1)$$

while for $\lambda = 1$ the conforming spinors are

$$\Phi = \frac{C}{2} \begin{pmatrix} \cos \frac{\vartheta}{2} - i \sin \frac{\vartheta}{2} \\ \cos \frac{\vartheta}{2} + i \sin \frac{\vartheta}{2} \end{pmatrix} e^{i\varphi/2}, \Phi = \frac{C}{2} \begin{pmatrix} -\cos \frac{\vartheta}{2} + i \sin \frac{\vartheta}{2} \\ \cos \frac{\vartheta}{2} + i \sin \frac{\vartheta}{2} \end{pmatrix} e^{-i\varphi/2} \quad (C.2)$$

with the coefficient $C = \sqrt{1/(2\pi)}$.

14 Appendix D: Condition $\sigma_2 \Phi \approx \sin \vartheta \Phi$

Let us compute spinor $\Phi_0 = (\sigma_2 - \sin \vartheta) \Phi$ where, for example, Φ is the first spinor in (C.1). We obtain

$$\Phi_0 = \sqrt{\frac{1}{8\pi}} \left[-\sin \frac{\vartheta}{2} (2 + \cos \vartheta) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \cos \frac{\vartheta}{2} (2 - \cos \vartheta) \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right] e^{i\varphi/2}, \quad (D.1)$$

so that both components of Φ_0 are modulo

$$\sqrt{\frac{1 + 3 \sin^2 \vartheta}{8\pi}} \leq \sqrt{\frac{1}{2\pi}} \approx 0.4 < 1,$$

i. e. condition $\sigma_2 \Phi \approx \sin \vartheta \Phi$ is good enough fulfilled. This holds true for all the eigen-spinors of \mathcal{D}_0 , or, more generally, of \mathcal{D}_n which we shall not dwell upon.

15 Appendix E: More General Ansatz

As was remarked in Subsection 8.1, there can be a more general ansatz than (8.1) for finding the confining solutions, namely it is

$$A = r^{\mu_a} \alpha^a \lambda_a dt + A_r dr + A_\vartheta d\vartheta + r^{\nu_a} \beta^a \lambda_a d\varphi, \quad (E.1)$$

with $A_\vartheta = r^{\rho_a} \gamma^a \lambda_a$ and arbitrary real constants $\alpha^a, \beta^a, \gamma^a$. Under this situation the Lorentz condition (B.4) for the given ansatz entails $\cot \vartheta r^{\rho_a} \gamma^a \lambda_a + \partial_r(r^2 A_r) = 0$ wherefrom it follows

$$A_r = \frac{C}{r^2} - \frac{\cot \vartheta r^{\rho_a-1}}{\rho_a + 1} \gamma^a \lambda_a \quad (E.2)$$

with a constant matrix C belonging to SU(3)-Lie algebra. Then we can see that it should put $C = \gamma^a = 0$ or else A_r will not be spherically symmetric and the confining one where only the powers of r equal to ± 1 are admissible. As a result, we come to the conclusion that one should put $A_r = A_\vartheta = 0$ in (E.1).

After this we have

$$\begin{aligned} dA &= -\mu_a r^{\mu_a-1} \alpha^a \lambda_a dt \wedge dr + \nu_a r^{\nu_a-1} \beta^a \lambda_a dr \wedge d\varphi, \\ F &= dA + gA \wedge A = dA + g\alpha^a \beta^b r^{\mu_a+\nu_b} [\lambda_a, \lambda_b] dt \wedge d\varphi \end{aligned} \quad (E.3)$$

and, with the help of (A.6), we obtain

$$\begin{aligned} *F &= \mu_a r^{\mu_a+1} \alpha^a \lambda_a \sin \vartheta d\vartheta \wedge d\varphi - \frac{\nu_a r^{\nu_a-1} \beta^a \lambda_a}{\sin \vartheta} dt \wedge d\vartheta - \\ &\quad g r^{\mu_a+\nu_b} \alpha^a \beta^b \frac{[\lambda_a, \lambda_b]}{\sin \vartheta} dr \wedge d\vartheta, \end{aligned} \quad (E.4)$$

$$d * F = \mu_a(\mu_a + 1)r^{\mu_a}\alpha^a\lambda_a \sin \vartheta dr \wedge d\vartheta \wedge d\varphi + \frac{\nu_a(\nu_a - 1)r^{\nu_a}\beta^a\lambda_a}{\sin \vartheta} dt \wedge dr \wedge d\vartheta, \quad (E.5)$$

so the Yang-Mills equations (B.3) (at $J = 0$) turn into the system

$$\mu_a(\mu_a + 1)r^{\mu_a}\alpha^a\lambda_a \sin^2 \vartheta = -g^2 r^{\mu_a + \nu_b + \nu_c} \alpha^a \beta^b \beta^c [[\lambda_a, \lambda_b], \lambda_c], \quad (E.6)$$

$$\nu_a(\nu_a - 1)r^{\nu_a - 2} \beta^a \lambda_a = -g^2 r^{\mu_a + \nu_b + \mu_c} \alpha^a \beta^b \alpha^c [[\lambda_a, \lambda_b], \lambda_c], \quad (E.7)$$

$$g(\mu_a - \mu_b)r^{\mu_a + \mu_b} \alpha^a \alpha^b [\lambda_a, \lambda_b] = 0, \quad g(\nu_a - \nu_b)r^{\nu_a + \nu_b} \beta^a \beta^b [\lambda_a, \lambda_b] = 0, \quad a < b, \quad . \quad (E.8)$$

Then, obviously, in the left-hand sides of (E.6–E.7) we should put all $\mu_a = -1$ and all $\nu_a = 1$, respectively, for obtaining the confining solutions while the right-hand sides of (E.6–E.7) are identically equal to zero only if $\lambda_{a,b,c} = \lambda_{3,8}$. Under the circumstances the equations (E.8) are satisfied automatically and we come to the same conclusions about uniqueness of the confining solutions as in Subsection 8.1.

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